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Strain gradient torsional vibration analysis of micro/nano rods

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ABSTRACT

Fabrication, characterization and application of micro-/nano-rods/wires are among the hottest topics in materials science and applied physics. Micro-/nano-rod-based structures and devices are developed for a wide-ranging use in various fields of micro-/nanoscience (e.g. biology, electronics, medicine, optics, optoelectronics, photonics and sensors). It is well known that the structure and properties of micro/nano rods depend greatly on their environment of application. Therefore, in this paper, torsional vibration of microbars is formulated based on the strain gradient theory to study the vibrational behavior at micro/nano scale. The strain gradient theory is a non-classical theory capable of capturing the size-effects. The governing equation and both the classical and the non-classical boundary conditions are derived employing the Hamilton's principle. In the free-vibration case, the characteristic equation is derived and solved analytically. The torsional free-vibration behavior of a fixed-fixed strain gradient microbar is investigated and the results are compared to those evaluated by the classical and modified couple stress theories noted that the two latter theories are special cases of the strain gradient theory. The effects of the length and the radius of the micro rods on the various modes of torsional natural frequencies are investigated in detail. The results of this study can be useful in the design and analysis of the next generation micro-electro-mechanical-systems and nano-electro-mechanical-systems which uses the torsional vibration properties of the micro-/nano-rods.

Keywords: *Micro/nano rod; Strain gradient elasticity; Modified couple stress theory; Torsional vibration; Modenumber; Small scale.*

INTRODUCTION

With the rapid development of technology, micro rods and micro tubes with internal diameters smaller than 1 mm are often used in micro- and nano-electro-mechanical systems (MEMS and NEMS), such as those employed in atomic force microscopes [1], sensors [2], actuators [3], fluid and mass transport [4-6].

Moreover, understanding and controlling the motion of small particles is critical for micro- and nanoassembly, microfluidics, including biological and colloidal science applications, chemical mechanical polishing, and xerographic processes [7].

The micro/nano solid/hollow rods subjected to torsional loads and torsional displacements are widely used in various kinds of MEMS/NEMS such as micro-gyroscopes [8,9], torsional microscanners [10], torsional micromirrors [9,11,12] and torsional spring in NEMS oscillators [13]. Hence, the accurate modeling of the static and dynamic torsional behavior of micro/nano bars seems to be crucial in studying the mechanical behaviors of these micro/nano systems.

Since the classical continuum mechanics is incapable of capturing the size effect and consequently unable to predict and interpret the size-dependent static and vibration behavior observed in micro-scaled structures, during past years, some non-classical continuum theories such as the nonlocal, strain gradient and couple stress theories have been introduced, developed and employed to investigate the micro-scaled structures. In these non-classical theories, some material parameters are considered in addition to the two classical parameters, elastic modulus and Poisson ratio, which enable these theories to capture the size-dependency. For example, in the modified couple stress theory, due to the micro structure rotation gradient, an additional length scale parameter is considered while in the strain gradient theory, there exist three additional length scale parameters corresponding to the micro structure rotation gradient, the micro structure dilatation gradient and the micro structure stretch gradient. Due to the lack of internal material length scale parameters, conventional strain-based mechanics theories fail to characterize those size effects phenomenon when the structural size is in micron- and sub-micron-scale. However, these size dependences can be successfully modeled by employing higher-order continuum theory, in which constitutive equations introduce additional material length scale parameters in addition to classical material parameters.

As a higher-order continuum theory, the classical couple stress elasticity theory was originated by Mindlin and others including Toupin and Koiter in 1960s and contains four material constants (two classical and two additional) for isotropic elastic materials [14-17]. Some related research works had been performed to model the static and dynamic problems based on the classical couple stress theory [18,19]. In 1994, Fleck and Hutchinson extended and reformulated the classical couple stress theory and renamed it as the strain gradient theory, in which for homogeneous isotropic and incompressible materials, two additional higher-order material length scale parameters are introduced for couple stress theory and three additional higher-order material length scale parameters are introduced for stretch and rotation gradient theory [20-23]. Recently, a modified couple stress theory for elasticity had been elaborated by Yang et al. in 2002, in which constitutive equations involve only one additional internal material length scale parameter besides two classical material constants [24]. This theory had been applied to the analysis of many boundary value problems successfully [24, 26]. In 2003, Lam et al. proposed a modified strain gradient elasticity theory in which a new additional equilibrium equations to govern the behavior of higher-order stresses, the equilibrium of moments of couples is introduced, in addition to the classical equilibrium equations of forces and moments of forces [27]. Moreover, there are only three independent higher-order materials length scale parameters for isotropic linear elastic materials in the present theory. Then an elastic bending theory for thin plane-strain beams was developed. Static bending solutions for cantilever beams were derived based on the new higher-order bending theory and the constitutive equations were expressed as the moment and higher-order moment in terms of the curvature and the curvature gradients [27]. In a similar way utilized by Yang et al. [24] for the modification of the couple stress theory, Lam et al. [27] introduced a modified strain gradient theory, which reduces in a special case to the modified couple stress theory. Henceforth, when the strain gradient theory is used in this text, it refers to the version of the theory presented by Lam et al. [27]. In the strain gradient theory, there exist three length scale parameters corresponding to the micro structure rotation gradient, the micro structure dilatation gradient and the micro structure stretch gradient.

In studies associated with the strain gradient theory, for numerical evaluations, the

researchers usually consider these three length scale parameters to be the same and indeed equal to the length scale parameter used in the modified couple stress theory [28, 29]. In order to determine the length scale parameter for a specific material, some typical experiments such as micro-bend test, micro-torsion test and specially micro/nano indentation test can be carried out. As an example, according to the micro-torsion test of thin copper wire [30], the copper length scale parameter has been reported 4 μm. Also, based on the micro-bend test of thin nickel and epoxy beams, the length scale parameter for nickel and epoxy has been estimated 5 μm [31] and 17.6 μm [27], respectively.

Recently, torsional vibration of nanorods is presented with nonlocal elasticity by Narendar [32]. In that work it was found that the strong effect of the nonlocal scale leads to substantially different behaviors of nanorods from those of macroscopic rods. Nonlocal rod model was developed for torsional vibration of nanorods. It was found that the torsional vibration frequencies were highly over estimated by the classical rod model because of ignoring the effect of small length scale. Aydogdu [33] investigated free axial vibration of uniform nanorods using nonlocal continuum theory of Eringen. The results showed that the nonlocal rod model overestimates the natural frequencies of the nanorod with respect to the classical one. Narendar and Gopalakrishnan [34] also studied longitudinal vibration of nanorods using a nonlocal bar model. The obtained results indicated that small scale parameter of the nonlocal model causes a certain band gap region in the longitudinal wave mode such that no wave propagation would occur. This issue was shown in the illustrated spectrum curves, as the regions, where the wave number tends to an infinite value. Narendar and Gopalakrishnan [35], studied the ultrasonic wave dispersion characteristics of a nanorods based on nonlocal strain gradient models (both second and fourth order). They derived the explicit expressions for wave numbers and the wave speeds of the nanorod. Their analysis showed that the fourth order strain gradient model gives approximate results over the second order strain gradient model for dynamic analysis. The second order strain gradient model gives a critical wave number at certain wave frequency, where the wave speeds are zero. They explained the dynamic response behavior of nanorods based on both the strain gradient models.

More recently, Narendar [36], studied the terahertz wave dispersion characteristics of nanorods based on the nonlocal elasticity incorporating the lateral inertia under the umbrella of continuum mechanics theory. He has shown that, the unstable second order strain gradient model can be replaced by considering the inertia gradient terms in the formulations. The effects of both the nonlocal scale and the diameter of the nanorod on spectrum curves are highlighted in that work [36]. There are some works by the authors related to the field of MEMS/NEMS [37-45].

In this paper, torsional vibration of micro rods is formulated based on the strain gradient theory. The Hamilton principle is utilized to obtain the governing equation and both classical and non-classical boundary conditions. Afterward, a closed-form analytical solution is derived for torsional vibration of micro rods. As a case study, the free-vibration torsional behavior of a fixed-fixed micro rod modeled by the strain gradient theory is presented and the results of the current model are compared to those of the modified couple stress theory and classical theory.

A REVIEW OF STRAIN GRADIENT THEORY OF ELASTICITY

A modified strain gradient elasticity theory was proposed by Lam et al. [27], in which a new additional equilibrium equations to govern the behavior of higher-order stresses, the equilibrium of moments of couples is introduced, in additional to the classical equilibrium equations of forces and moments of forces [27, 46]. In the constitutive equations of this theory, there are only three independent higher-order materials length scale parameters in addition to two classical material parameters for isotropic linear elastic materials. Then the strain energy Π_{se} in a deformed isotropic linear elastic material occupying region Ω is given by

$$\Pi_{se} = \int_{\Omega} \bar{u} \, d\Omega = \frac{1}{2} \int_{\Omega} \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^S\chi_{ij}^S \right) d\Omega \quad (1)$$

Where the deformation measures are defined as follows,

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \qquad (2)$$

$$\varepsilon_{mm,i} = \gamma_i = \partial_i \varepsilon_{mm} \qquad (3)$$

$$\eta^{(1)}_{ijk} = \frac{1}{3}\left(\partial_i \varepsilon_{jk} + \partial_j \varepsilon_{ki} + \partial_k \varepsilon_{ij}\right) - \frac{1}{15}\delta_{ij}\left(\partial_k \varepsilon_{mm} + 2\partial_m \varepsilon_{mk}\right)$$
$$-\frac{1}{15}\left[\delta_{jk}\left(\partial_i \varepsilon_{mm} + 2\partial_m \varepsilon_{mi}\right) + \delta_{ki}\left(\partial_j \varepsilon_{mm} + 2\partial_m \varepsilon_{mj}\right)\right] \qquad (4)$$

$$\chi^s_{ij} = \frac{1}{2}\left(\partial_i \varphi_j + \partial_j \varphi_i\right) \qquad (5)$$

$$\varphi_i = \frac{1}{2}\,\text{curl}(\mathbf{u}) \qquad (6)$$

Where u_i , γ_i σφτ and θ_i denote the components of the displacement vector u, the dilatation gradient vector γ, and the infinitesimal rotation vector φ. Also, the components of the strain tensor ε, the deviatoric stretch gradient tensor $\eta^{(1)}$, and the symmetric part of the rotation gradient tensor χ^S are represented by ε_{ij} , $\eta^{(1)}{}_{ijk}$ and χ^S_{ij} . The parameters which are obtained by differentiating the strain energy density with respect to kinematics parameters ε, γ, $\eta^{(1)}$ and $\chi^S_{}$ are, respectively, symbolized by σ, p, $\tau^{(1)}$ and m^S. The parameters p, $\tau^{(1)}$ and m^S are usually called the higher-order stresses. According to the constitutive equations for a linear isotropic elastic material, the components of the stresses are related to the kinematic parameters effective on $\bar{u}$ as follows:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij}' \qquad (7)$$

$$p_i = 2\mu l_0^2 \gamma_i \qquad (8)$$

$$\tau^{(1)}_{ijk} = 2\mu l_1^2 \eta^{(1)}_{ijk} \qquad (9)$$

$$m^S_{ij} = 2\mu l_2^2 \chi^S_{ij} \qquad (10)$$

where ε_{ij}' is deviatoric strain defined as

$$\varepsilon_{ij}' = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{mm} \qquad (11)$$

In the above equations, λ, μ are the Lame constants appearing in the constitutive equation of the classical stress σ. Also, the additional independent material length scale parameters appeared in the constitutive equations of higher order stresses are represented by l_1 , l_2 and l_3, respectively associated with dilatation gradients, deviatoric stretch gradients and rotation gradients. The Lame constants can be written in terms of the Young modulus E and the Poisson ratio v as

$$\lambda = \frac{vE}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)}. \qquad (12)$$

DYNAMIC ANALYSIS OF MICRO RODS BASED ON STRAIN GRADIENT THEORY OF ELASTICITY

Governing equation and boundary conditions for torsional vibration

In this section, the governing equation and both classical and non-classical boundary conditions of torsional vibration of a micro bar based on strain gradient theory are derived.

Consider a straight micro rod with lengtH L_{mr} with coordinate system as shown in Figure 1a and its cross-section with radius is shown in Figure 1b. For torsion of the micro rod depicted in Figure 1c, the components of the displacement vector u are expressed as [47].

$$u_1(z,t) = -y\theta(z,t) \qquad (13)$$

$$u_2(z,t) = x\theta(z,t) \qquad (14)$$

$$u_3(z,t) = 0 \qquad (15)$$

where u_1, u_2 and u_3 denote the displacement along x, y and z axes, respectively; where z represents the coordinate along the micro rod length. Also, θ stands for rotation angle of the micro rod about $z-$axis and u=u_1i+u_2j+u_3k. By substitution of Eqs. (13)-(15) into Eq. (2), the components of the strain tensor   are obtained as

$$\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{21} = \varepsilon_{22} = \varepsilon_{33} = 0 \qquad (16)$$

$$\varepsilon_{13} = \varepsilon_{31} = -\frac{1}{2} y \frac{\partial \theta}{\partial z} \qquad (17)$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2} x \frac{\partial \theta}{\partial z} \qquad (18)$$

Substituting Eqs.(16)-(18) into Eqs.(3) and (4), the components of the dilatation gradient vector $\boldsymbol{\gamma}$ and the nonzero components of the deviatoric stretch gradient tensor $\boldsymbol{\eta}^{(1)}$ are respectively determined as

$$\gamma_1 = \gamma_2 = \gamma_3 = 0 \qquad (19)$$

$$\eta_{111}^{(1)} = \frac{1}{5} y \frac{\partial^2 \theta}{\partial z^2}$$

$$\eta_{112}^{(1)} = \eta_{121}^{(1)} = \eta_{211}^{(1)} = -\frac{1}{15} x \frac{\partial^2 \theta}{\partial z^2}$$

$$\eta_{212}^{(1)} = \eta_{221}^{(1)} = \eta_{122}^{(1)} = \frac{1}{15} y \frac{\partial^2 \theta}{\partial z^2}$$

$$\eta_{222}^{(1)} = -\frac{1}{5} x \frac{\partial^2 \theta}{\partial z^2}$$

$$\eta_{313}^{(1)} = \eta_{133}^{(1)} = \eta_{331}^{(1)} = -\frac{4}{15} y \frac{\partial^2 \theta}{\partial z^2}$$

$$\eta_{323}^{(1)} = \eta_{233}^{(1)} = \eta_{332}^{(1)} = \frac{4}{15} x \frac{\partial^2 \theta}{\partial z^2} \qquad (20)$$

The components of the rotation vector $\boldsymbol{\varphi}$ are derived by substitution of Eqs. (13)-(15) into Eq. (6), as follows:

$$\varphi_1 = -\frac{1}{2} x \theta(z,t), \ \ \varphi_2 = -\frac{1}{2} y \theta(z,t), \ \ \varphi_3 = \theta(z,t) \quad (21)$$

Having the components of the rotation vector $\boldsymbol{\varphi}$ in hand, one can obtain the non-zero components of the symmetric part of the rotation gradient tensor $\boldsymbol{\chi}^S$ introduced in Eq. (5) as

$$\chi_{11}^S = \chi_{22}^S = -\frac{1}{2} \frac{\partial \theta}{\partial z}$$

$$\chi_{33}^S = \frac{\partial \theta}{\partial z}$$

$$\chi_{13}^S = \chi_{31}^S = -\frac{1}{4} x \frac{\partial^2 \theta}{\partial z^2}$$

$$\chi_{23}^S = \chi_{32}^S = -\frac{1}{4} y \frac{\partial^2 \theta}{\partial z^2} \qquad (22)$$

Similarly, employing the components $\boldsymbol{\varepsilon}$, $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\chi}^S$ obtained before and considering Eqs. (7), (9), and (10), the non-zero components of the stress tensor $\boldsymbol{\sigma}$ and higher-order stress tensors, $\boldsymbol{\tau}^{(1)}$ and $\mathbf{m}^S$ are determined as

$$\sigma_{13} = \sigma_{31} = -\mu y \frac{\partial \theta}{\partial z}$$

$$\sigma_{23} = \sigma_{32} = \mu x \frac{\partial \theta}{\partial z} \qquad (23)$$

$$\tau_{111}^{(1)} = \frac{2}{5} \mu l_1^2 y \frac{\partial^2 \theta}{\partial z^2}$$

$$\tau_{112}^{(1)} = \tau_{121}^{(1)} = \tau_{211}^{(1)} = -\frac{2}{15} \mu l_1^2 x \frac{\partial^2 \theta}{\partial z^2}$$

$$\tau_{212}^{(1)} = \tau_{221}^{(1)} = \tau_{122}^{(1)} = \frac{2}{15} \mu l_1^2 y \frac{\partial^2 \theta}{\partial z^2}$$

$$\tau_{222}^{(1)} = -\frac{2}{5} \mu l_1^2 x \frac{\partial^2 \theta}{\partial z^2}$$

$$\tau_{313}^{(1)} = \tau_{133}^{(1)} = \tau_{331}^{(1)} = -\frac{8}{15} \mu l_1^2 y \frac{\partial^2 \theta}{\partial z^2}$$

$$\tau_{323}^{(1)} = \tau_{233}^{(1)} = \tau_{332}^{(1)} = \frac{2}{15} \mu l_1^2 x \frac{\partial^2 \theta}{\partial z^2} \qquad (24)$$

$$m_{11}^S = m_{22}^S = -\mu l_2^2 \frac{\partial \theta}{\partial z}$$

$$m_{33}^S = 2\mu l_2^2 \frac{\partial \theta}{\partial z}$$

$$m_{13}^S = m_{31}^S = -\frac{1}{2} \mu l_2^2 x \frac{\partial^2 \theta}{\partial z^2}$$

$$m_{23}^S = m_{32}^S = -\frac{1}{2} \mu l_2^2 y \frac{\partial^2 \theta}{\partial z^2} \qquad (25)$$

It is noted that according to Eq. (8), since all components of the dilatation gradient vector are zero, i.e. $\gamma_i = 0$, all components of the higher-order stress vector **P** will be zero too. By substituting Eqs. (16)-(20), (22)-(25) into Eq.(1), the total potential energy of the micro rod Π_{se} is obtained as follows:

$$\Pi_{se} = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^S\chi_{ij}^S\right)dV = \frac{1}{2}\int_0^{L_{mr}}\int_A \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^S\chi_{ij}^S\right)dAdz$$

$$= \frac{1}{2}\int_0^{L_{mr}}\mu\left[\left(\int_A(x^2+y^2)dA + 3Al_2^2\right)\left(\frac{\partial\theta}{\partial z}\right)^2 + \left(\int_A(x^2+y^2)dA\right)\times\left(\frac{8}{15}l_1^2 + \frac{1}{4}l_2^2\right)\left(\frac{\partial^2\theta}{\partial z^2}\right)^2\right]dz \tag{26}$$

where V, A and L_{mr}, respectively denote the volume, cross-section area and length of the micro rod. Furthermore, $\int_A(x^2+y^2)dA$ is the polar area moment of inertia and later it will be denoted as J. The total kinetic energy of the micro rod Π_{ke} is obtained as

$$\Pi_{ke} = \frac{1}{2}\int_V \rho\left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right]dV = \frac{1}{2}\int_0^{L_{mr}}\int_A \rho\left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right]dAdz \tag{27}$$

$$= \frac{1}{2}\int_0^{L_{mr}}\rho J\left(\frac{\partial\theta}{\partial t}\right)^2 dz$$

here ρ is the density of the micro rod.

The variation of the work done by external classical and higher-order torques, Π_{wd}, can be introduced as

$$\delta\Pi_{wd} = \int_0^{l_{mr}}T(z,t)\delta\theta dz + \mathrm{T}^C\delta\theta\Big|_{z=0}^{z=L_{mr}} + \mathrm{T}^H\delta\frac{\partial\theta}{\partial z}\Big|_{z=0}^{z=L_{mr}} \tag{28}$$

where $T(z,t)$ represents the distributed torque per unit length about z-axis and T^C and T^H denote respectively, the classical and higher order torques acting on the end sections of the micro rod.

The governing torsional vibration equation and boundary conditions of micro rods based on strain gradient theory are obtained by utilizing the Hamilton principle. The Hamilton principle yields

$$\int_{t_1}^{t_2}\left(\delta\Pi_{ke} - \delta\Pi_{se} + \delta\Pi_{wd}\right)dt = 0 \tag{29}$$

This implies

$$\int_{t_1}^{t_2}\int_0^{L_{mr}}\left[\rho J\frac{\partial\theta}{\partial t}\delta\frac{\partial\theta}{\partial t} - \mu\left(J + 3Al_2^2\right)\frac{\partial\theta}{\partial z}\delta\frac{\partial\theta}{\partial z} - \mu J\left(\frac{8}{15}l_1^2 + \frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}\delta\frac{\partial^2\theta}{\partial z^2} - T(z,t)\delta\theta\right]dzdt$$

$$-\int_{t_1}^{t_2}\left[\mathrm{T}^C\delta\theta\Big|_{z=0}^{z=L_{mr}} + \mathrm{T}^H\delta\frac{\partial\theta}{\partial z}\Big|_{z=0}^{z=L_{mr}}\right]dt = 0 \tag{30}$$

Using integration by parts, this equation becomes

$$\int_{t_1}^{t_2}\int_0^{L_{mr}}\{-\frac{\partial}{\partial z}\left[\mu\left(J+3Al_2^2\right)\frac{\partial\theta}{\partial z}\right]+\frac{\partial^2}{\partial z^2}\left[\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}\right]+\frac{\partial}{\partial t}\left(\rho J\frac{\partial\theta}{\partial t}\right)-T(z,t)\}\delta\theta\;dz\;dt$$

$$+\int_{t_1}^{t_2}\left[\frac{\partial}{\partial z}\left(\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}\right)-\left(\mu\left(J+3Al_2^2\right)\frac{\partial\theta}{\partial z}\right)+\mathrm{T}^C\right]\delta\theta\Bigg|_{z=0}^{z=L_{mr}}dt$$

$$+\int_{t_1}^{t_2}\left[\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}-\mathrm{T}^H\right]\delta\frac{\partial\theta}{\partial z}\Bigg|_{z=0}^{z=L_{mr}}dt+\int_0^{L_{mr}}\rho J\frac{\partial\theta}{\partial t}\delta\theta\Bigg|_{t=t_1}^{t=t_2}dz=0$$

(31)

Based on the Lagrangian equations, the governing torsional vibration equation, boundary and initial conditions of micro rods based on strain gradient theory are obtained as:

Governing Equation
δθ:

$$-\frac{\partial}{\partial z}\left[\mu\left(J+3Al_2^2\right)\frac{\partial\theta}{\partial z}\right]+\frac{\partial^2}{\partial z^2}\left[\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}\right]+\frac{\partial}{\partial t}\left(\rho J\frac{\partial\theta}{\partial t}\right)-T(z,t)=0$$

(32)

Boundary Conditions
δθ:

$$\left[\frac{\partial}{\partial z}\left(\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}\right)-\left(\mu\left(J+3Al_2^2\right)\frac{\partial\theta}{\partial z}\right)+\mathrm{T}^C\right]\Bigg|_{z=0}^{z=L_{mr}}=0$$

(33)

or

$$\delta\theta\Big|_{z=0}^{z=L_{mr}}$$

(34)

$$\delta\frac{\partial\theta}{\partial z}:$$

$$\left[\mu J\left(\frac{8}{15}l_1^2+\frac{1}{4}l_2^2\right)\frac{\partial^2\theta}{\partial z^2}-\mathrm{T}^H\right]\Bigg|_{z=0}^{z=L_{mr}}$$

(35)

or

$$\delta\frac{\partial\theta}{\partial z}\Bigg|_{z=0}^{z=L_{mr}}$$

(36)

Initial Conditions

δθ:

$$\rho J\frac{\partial\theta}{\partial t}\Bigg|_{t=t_1}^{t=t_2}=0$$

(37)

or

$$\delta\theta\Big|_{t=t_1}^{t=t_2}=0$$

(38)

Here Eq. (32) represents the governing equation of torsion of micro rods based on strain gradient theory of elasticity while Eqs. (33)-(36) refer to corresponding classical and non-classical boundary conditions, respectively and Eqs. (37) and (38) refer to the initial conditions.

Now by substituting $l_1=0$ and $l_2=l$ in Eqs. (32)-(38), reduce to the governing equation and boundary an initial conditions of torsional vibration of a micro rod modeled on the basis of the modified couple stress theory (MCST) presented in ref. [48]. It is noted that the modified couple stress theory is a special case of the strain gradient theory in a way that by letting $l_0=l_1=0$ and $l_2=l$ in strain gradient theory, the formulation of the modified couple stress theory can be achieved. Furthermore, letting $l_1=l_2=0$, Eqs. (32)-(38) reduce to governing equation, boundary and initial conditions of torsion of a classical rod presented in ref. [49].

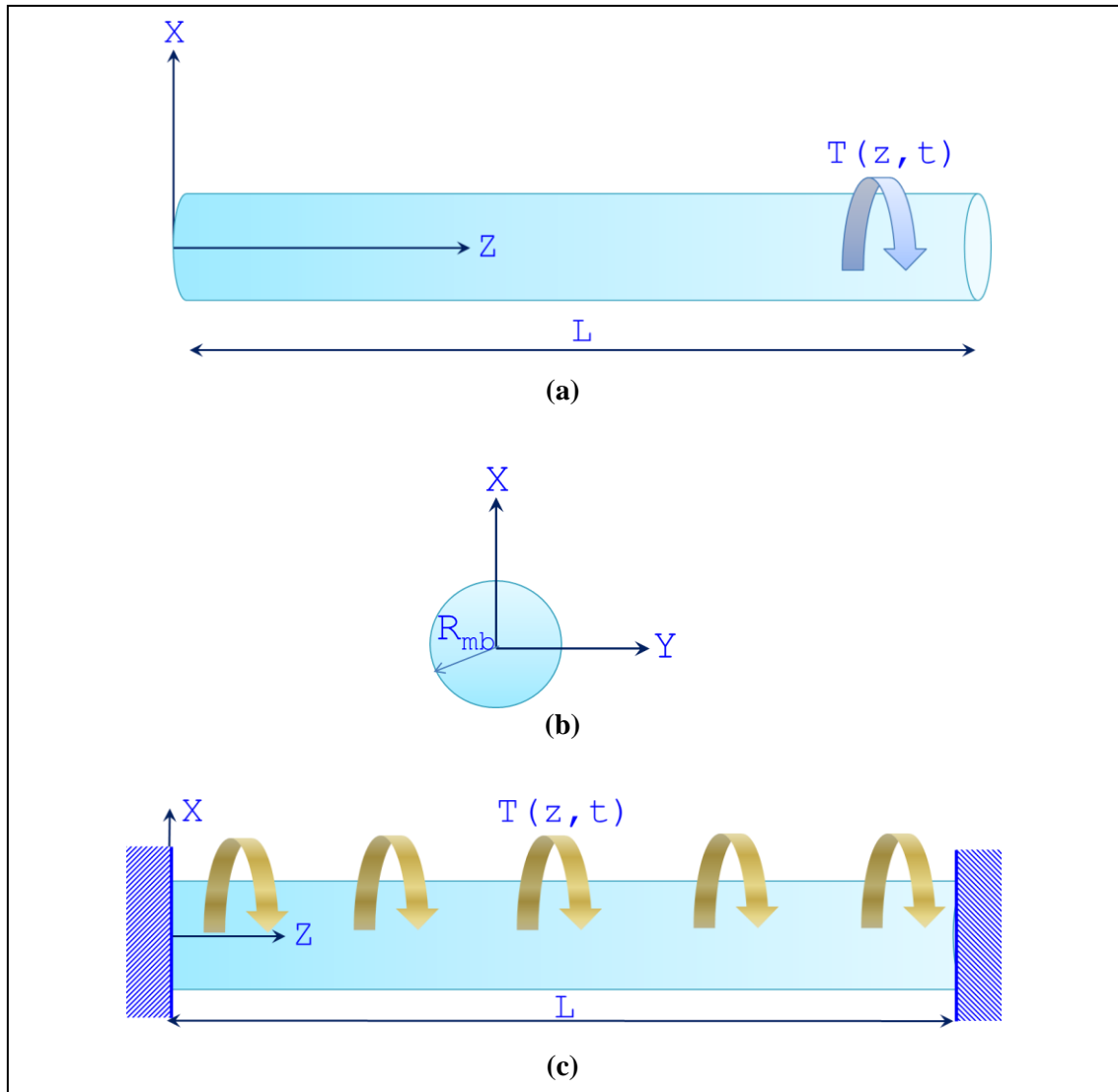**Fig. 1.** (a) Schematic of the micro rod showing cooerinate system, length and the applied torque, (b) cross section of the micro rod showing the radius and the coodrinate system and (c) the fixed-fixed micro bar under torsional vibration.

Solution of the governing equation: Free torsional vibration analysis

For constant values for μ, J, A, ρ, l_1 and l_2, considering $T(z,t) = 0$ the governing equation of torsion of micro rods based on strain gradient theory of elasticity can be written as:

$$-\mu\left(J + 3Al_2^2\right)\frac{\partial^2\theta}{\partial z^2} + \mu J\left(\frac{8}{15}l_1^2 + \frac{1}{4}l_2^2\right)\frac{\partial^4\theta}{\partial z^4} + \rho J\frac{\partial^2\theta}{\partial t^2} = 0 \quad (39)$$

For free-vibration behavior of a micro rod, one can assume a harmonic response for Eq. (39) as

$$\theta(z,t) = \Theta(z)e^{j\omega t} \quad (40)$$

here ω denote the torsional natural frequency of the micro rod and $j = \sqrt{-1}$. By substituting Eq. (40) into Eq. (39), the following is obtained

$$-\mu\left(J + 3Al_2^2\right)\frac{d^2\Theta(z)}{dz^2} + \mu J\left(\frac{8}{15}l_1^2 + \frac{1}{4}l_2^2\right)\frac{d^4\Theta(z)}{dz^4} - \rho J\omega^2\Theta(z) = 0 \quad (41)$$

In order to describe the size-dependent free torsional vibration of micro rods, a numerical example is presented here. Consider a fixed-fixed micro rod with length L_{mr} having circular cross-section and neglect the loading. For the aforementioned micro rod, the boundary conditions of Eq. (41) can be presented as

$$\Theta\big|_{z=0} = 0, \ \frac{d^2\Theta}{dz^2}\bigg|_{z=0} = 0, \ \Theta\big|_{z=L_{mr}} = 0, \ \frac{d^2\Theta}{dz^2}\bigg|_{z=L_{mr}} = 0 \qquad (42)$$

Assuming the solution of the Eq. (41) as $\Theta(z) = \tilde{\Theta} e^{j\xi z}$, here $\tilde{\Theta}$ is an unknown coefficient and $\xi = n\pi / L_{mr}$. Substituting it in Eq. (41) and solving for the torsional natural frequency of the micro rod as

$$\omega = \frac{\xi}{\sqrt{\rho J}} \sqrt{\mu\xi^2 \left(J + 3Al_2^2\right) + \mu J \left(\frac{8}{15}l_1^2 + \frac{1}{4}l_2^2\right)} \qquad (43)$$

By choosing $l_1 = 0$ and $l_2 = l$ in the above equation, the torsional natural frequency of a fixed–fixed micro rod modeled on the basis of the modified couple stress theory can be obtained. Furthermore, the natural torsional frequency of a classical fixed–fixed rod can be derived from Eq. (43) by letting $l_1 = l_2 = 0$.

RESULTS AND DISCUSSION

To illustrate the newly derived solution of the fixed-fixed micro rod based on strain gradient theory, some numerical results have been obtained and presented in this section, where the first five natural frequencies of a fixed-fixed micro rod given by the current strain gradient elasticity theory solution, modified couple stress theory solution and classical theory solution are shown. For the purpose of illustration, the rod considered here is taken to be made of epoxy and material properties used in the calculations are taken to be $E = 1.44\,GPa$, $v = 0.38$, and material density $\rho = 1000\,kg/m^3$. The length scale parameter for epoxy has been taken from ref. [27] as $17.6\,\mu m$.

From the curves in Figures 2a-2e, it can be seen that the first torsional natural frequencies predicted by the present strain gradient elastic rod theory are higher than that predicted by the classical beam theory and the modified couple stress theory. The torsional natural frequencies predicted by the modified couple stress theory are about 1.5 times lower than that predicted by the present strain gradient theory when the length of the micro rod is equal to 5 μm. It is also shown that the difference among the three sets of predicted values is diminishing when the radius of the rod becomes larger, thereby indicating that the size effect is only significant when the radius of the rod is comparable to the material length scale parameter. Compared to natural frequencies from modified couple stress theory, the natural frequencies from this strain gradient elasticity theory are larger and the size effects are reasonable that the strain gradient elasticity theory introduces additional dilatation gradient tensor and the deviatoric stretch gradient tensor in addition to the rotation gradient tensor.

The effect of length of the micro rod on the first torisonal natural frequencies can be seen from Figures 2a-2e. As the length of the micro rod increases from 5 μm to 100 μm, the difference between the present strain gradient elasticity and the modified couple stress theory becomes negligible even for higher radii of the micro rods. It can also be observed that the classical elasticity will not capture any size effects.

As we move from first torsional natural frequencies of micro rods to fifth torsional natural frequencies, it can be observed that the difference between the present strain gradient elasticity and the modified couple stress theory becomes significant and cannot be neglected for higher radii of the micro rods (see Figures 2a-2e, 3a-3e, 4a-4e, 5a-5e and 6a-6e).

It also confirms that the micro rods modelled by the strain gradient theory are stiffer than those modelled by the modified couple stress theory; noted that the modified couple stress theory predicts rods stiffer than the classical theory does. In addition, from Figures 2-6, it can be inferred that the difference between the results predicted by the strain gradient theory and those evaluated by the modified couple stress theory is greater for higher natural frequencies (higher modes) than for lower ones.
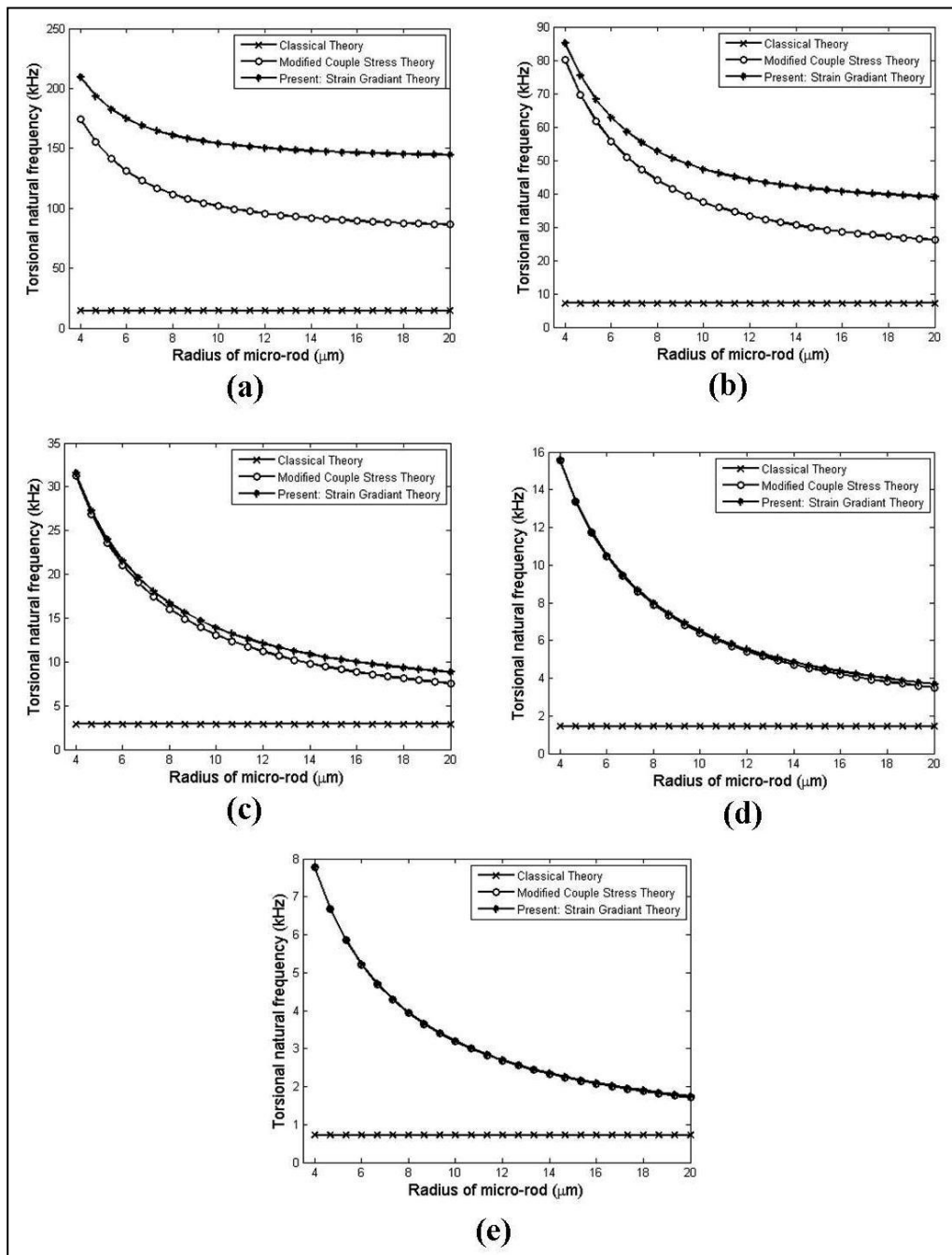
Fig. 2. Variation of first torsional natural frequencies of micro rod with radius for various lengths (a) L = 5 µm, (b) L = 10 µm, (c) L = 25 µm, (d) L = 50 µm, (e) L = 100 µm.

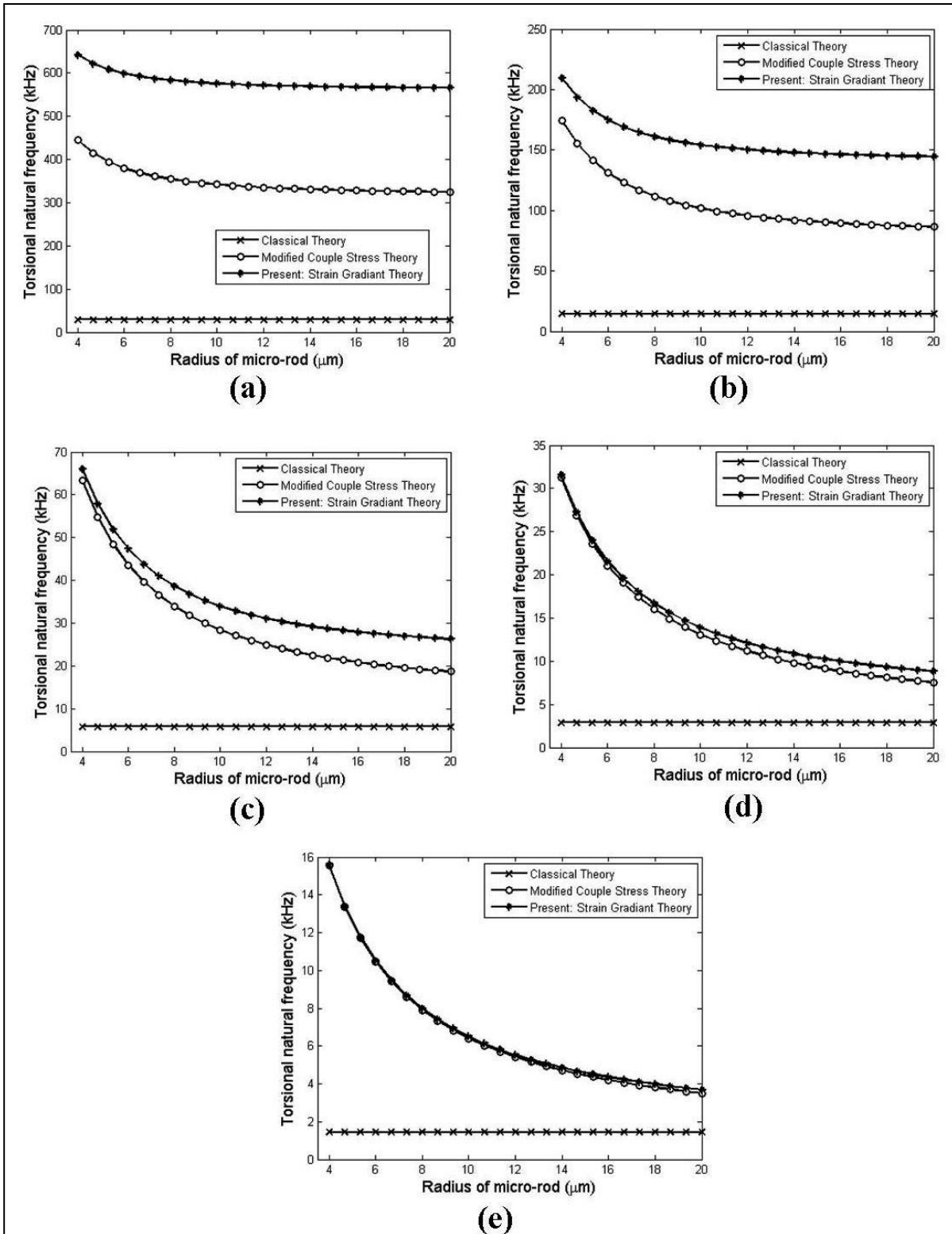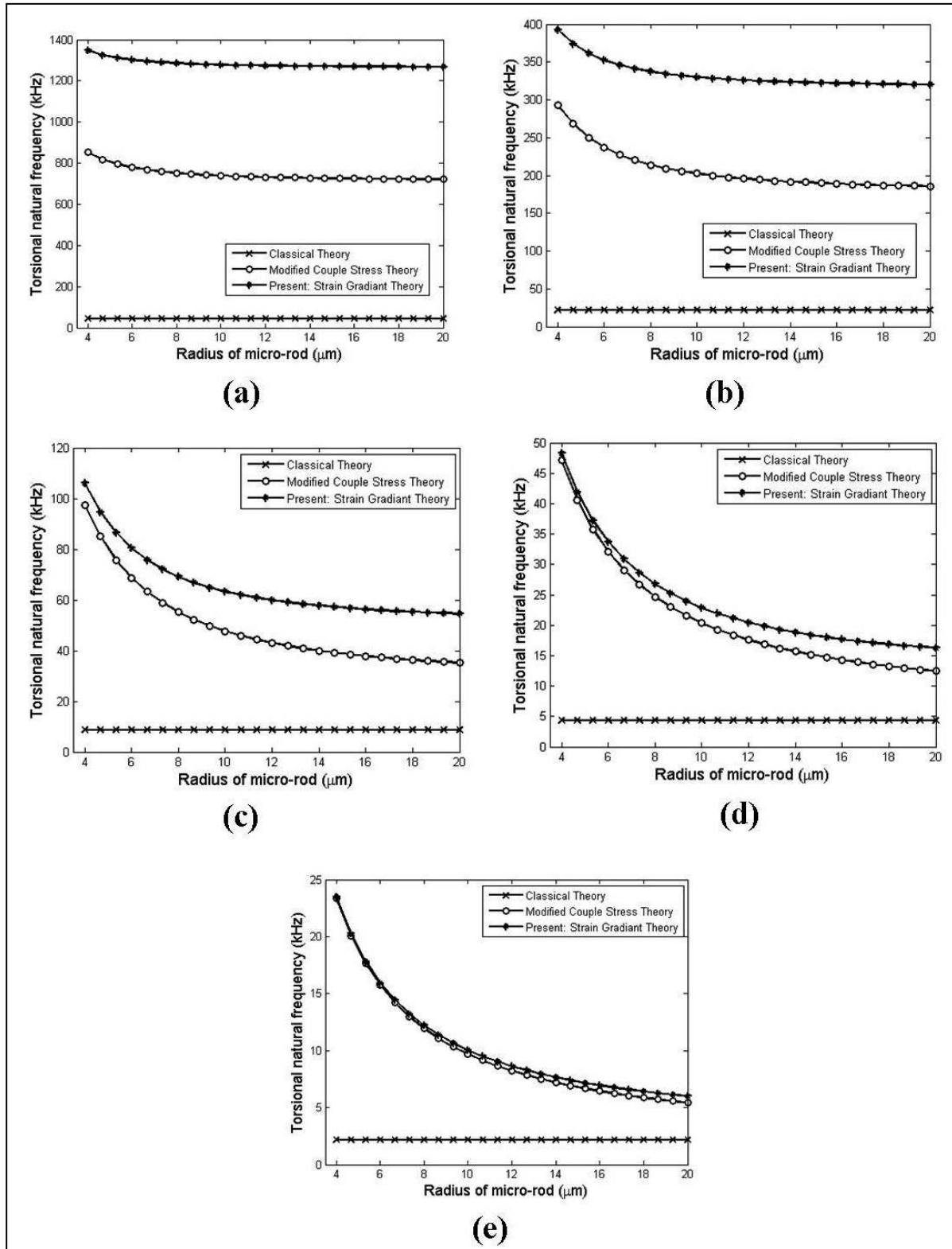**Fig. 3.** Variation of second torsional natural frequencies of micro rod with radius for various lengths (a) L = 5 µm, (b) L = 10 µm, (c) L = 25 µm, (d) L = 50 µm, (e) L = 100 µm.

Fig. 4. Variation of third torsional natural frequencies of micro rod with radius for various lengths (a) L = 5 µm, (b) L = 10 µm, (c) L = 25 µm, (d) L = 50 µm, (e) L = 100 µm.
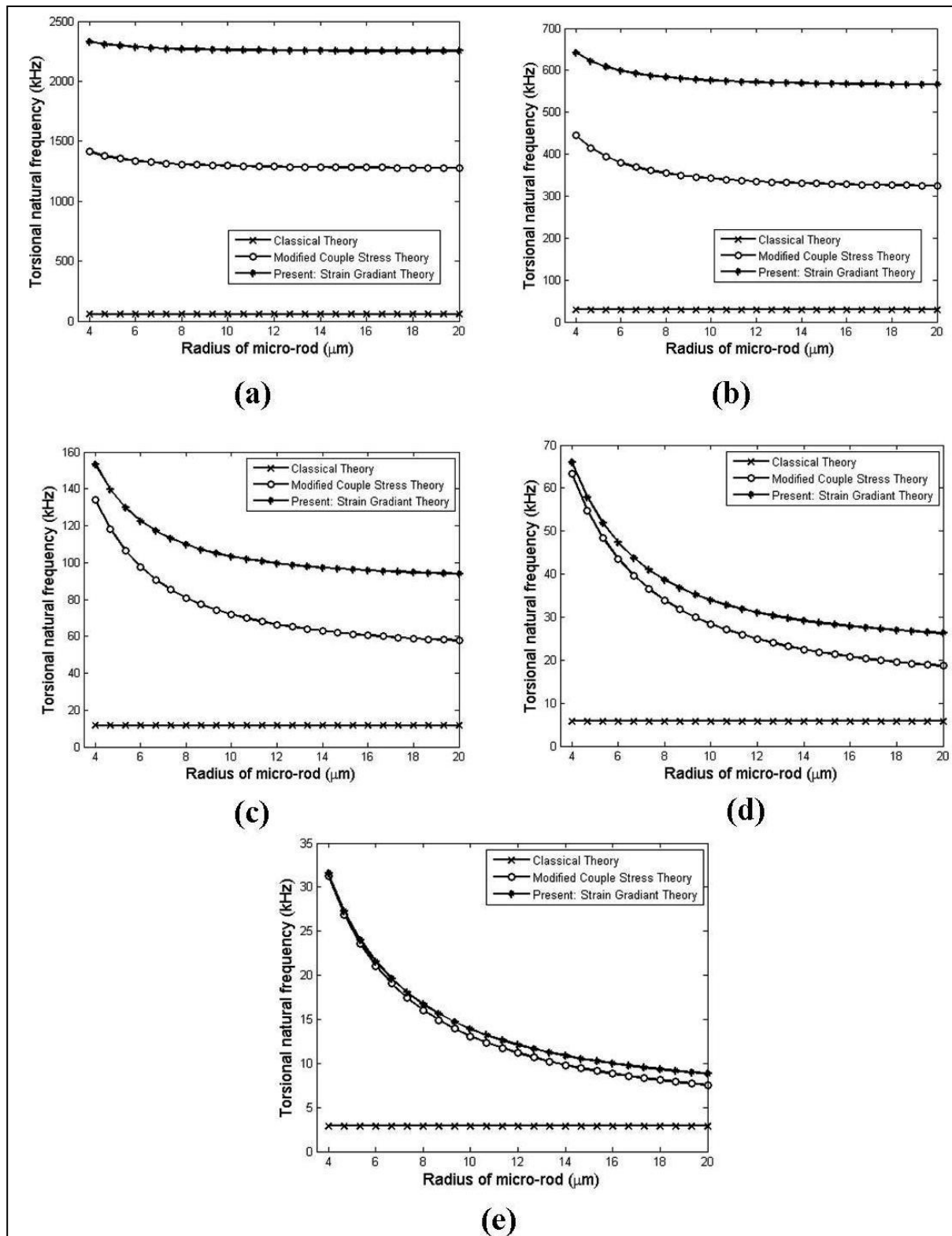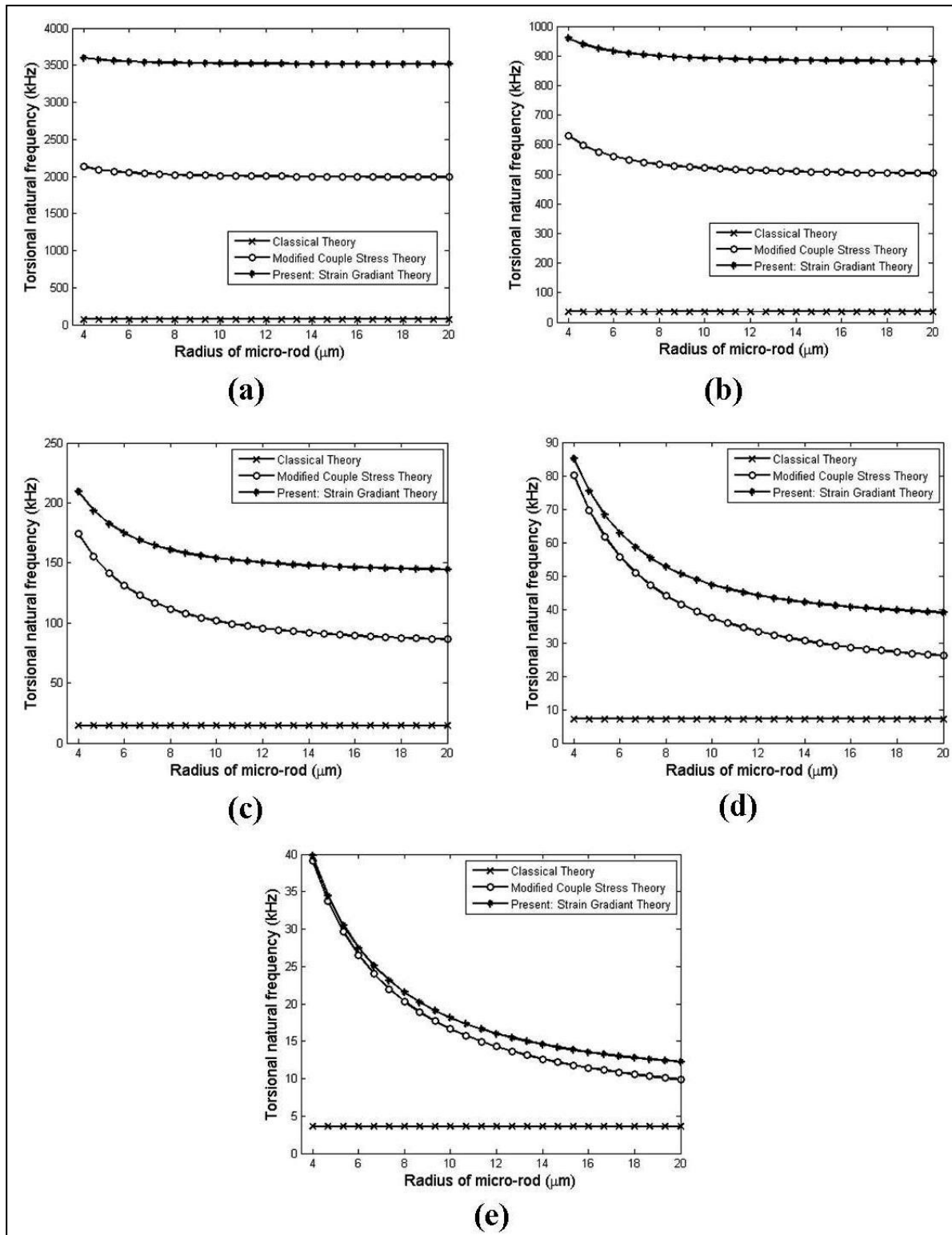
Fig. 5. Variation of fourth torsional natural frequencies of micro rod with radius for various lengths (a) L = 5 µm, (b) L = 10 µm, (c) L = 25 µm, (d) L = 50 µm, (e) L = 100 µm.

Fig. 6. Variation of fifth torsional natural frequencies of micro rod with radius for various lengths (a) L = 5 μm, (b) L = 10 μm, (c) L = 25 μm, (d) L = 50 μm, (e) L = 100 μm.

CONCLUSION

The torsional free-vibration behavior of a fixed-fixed micro rod based on strain gradient elasticity theory is investigated and the results are compared to those evaluated by the classical and the modified couple stress theories noted that the two latter theories are special cases of the strain gradient theory. In free-torsional-vibration case, the difference between the results of the strain gradient theory and those of the modified couple stress theory is greater for higher vibration-modes of micro rods than for lower ones. It has also been observed that the difference between the present strain gradient elasticity and the modified couple stress theory becomes negligible even for higher radii of the micro rods for lower lengths. The present results can be useful in the design and analysis of next generation micro-electro-mechanical-systems and nano-electro-mechanical-systems which uses the torsional vibration properties of the micro-/nano-rods.

REFERENCES

[1] Wang. W. L, Hu. S. J, (2005), Modal response and frequency shift of the cantilever in a noncontact atomic force microscope. *Appl. Phy. Let.* 87:183506.

[2] Zook. J. D, Burns. D. W, Guckel. H, Smegowsky. J. J, Englestad. R. L, Feng. Z, (1992), Characteristics of polysilicon resonant microbeams. *Sens. Act.* 35:51–59.

[3] Rezazadeh. G, Tahmasebi. A, Zubtsov. M, (2006), Application of piezoelectric layers in electrostatic MEM actuators: controlling of pull-in voltage, *J. Microsys. Tech.* 12:1163–1170.

[4] Peng. X. F, Peterson. G. P, Wang. B. X, (1994), Frictional flow characteristics of water flowing through rectangular microchannels, *Exp. Heat Tran.* 7:249–264.

[5] Sobhan. C. B, Garimella. S. V, (2001), A comparative analysis of studies on heat transfer and fluid flow in microchannels, Micro. *Thermo. Eng.* 5:293–311.

[6] Barlay Ergu. O, Sara. O. N, Yapıcı. S, Arzutug. M. E, (2009), Pressure drop and point mass transfer in a rectangular microchannel, Int. Comm. *Heat Mass Tran.* 36:618–623.

[7] Eglin. M, Eriksson. M. A, Carpick. R.W, (2006), Microparticle manipulation using inertial forces, *Appl. Phys. Lett.* 88:091913.

[8] Maenaka. K, Ioku. S, Sawai. N, Fujita. T, Takayama. Y, (2005), Design, fabrication and operation of MEMS gimbal gyroscope, *Sen. Act. A,* 121:6–15.

[9] Fujita. T, Maenaka. K, Takayama. Y, (2005), Dual-axis MEMS mirror for large deflection-angle using SU-8 soft torsion beam, *Sen. Act. A*, 121:16–21.

[10] Arslan. A, Brown. D, Davis. W, Holmstrom. S, Gokce. S. K, Urey. H, (2010), Comb-actuated resonant torsional microscanner with mechanical amplification, *J. Micro. Elect. Mech. Sys.* 19:936–943.

[11] Huang. J. M, Liu. A. Q, Deng. Z. L, Zhang. Q. X, (2006), A modeling and analysis of spring-shaped torsion micromirrors for low-voltage applications, *Int. J. Mech. Sci.*, 48:650–661.

[12] Zhang. X. M, Chau. F. S, Quan. C, Lam. Y. L, Liu A. Q, (2001), A study of the static characteristics of a torsional micromirror, Sens. *Act. A*, 90:73–81.

[13] Papadakis. S. J, Hall. A. R, Williams. P. A, Vicci. L, Falvo. M. R, Superfine. R, Washburn. S, (2004), Resonant oscillators with carbon-nanotube torsion springs, *Phy. Rev. Let.*, 93(14):146101.

[14] Mindlin. R. D, Tiersten. H. F, (1962), Effects of couple-stresses in linear elasticity, *Arch. Rat. Mech. Analysis*, 11(1):415–448.

[15] Mindlin. R. D, (1964), Micro-structure in linear elasticity, *Arch. Rat. Mech. Analysis*, 16(1):51–78.

[16] Mindlin. R. D, (1965), Second gradient of strain and surface-tension in linear elasticity, *Int. J. Solid. Struct.* 1(1):417–438.

[17] Toupin. R. A, (1962), Elastic materials with couple-stresses, *Arch. Rat. Mech. Analysis*, 11(1):385–414.

[18] Zhou. S. J, Li. Z. Q, (2001), Length scales in the static and dynamic torsion of a circular cylindrical micro-bar, *J. Shan. Univ. Tech.*, 31(5):401–407.

[19] Kang. X, Xi. W. X, (2007), Size effect on the dynamic characteristic of a micro beam based on cosserat theory, *J. Eng. Strength*, 29(1):1–4.

[20] Fleck. N. A, Hutchinson. J. W, (1993), Phenomenological theory for strain gradient effects in plasticity, *J. Mech. Phy. Solid*., 41(12):1825–1857.

[21] Fleck. N. A, Hutchinson. J. W, (1997), Strain gradient plasticity, *Adv. Appl. Mech.,* 33:296–358.

[22] Fleck. N. A, Hutchinson. J. W, (2001), A reformulation of strain gradient plasticity, *J. Mech. Phys. Solid*., 49(10):2245 – 2271.

[23] Smyshlyaev. V. P, Fleck. N. A, (1996), Role of strain gradients in the grain size effect for polycrystals, *J. Mech. Phys. Solid*., 44(4):465 – 495.

[24] Yang. F, Chong. A. C. M, Lam. D. C. C, (2002), Couple stress based strain gradient theory for elasticity, *Int. J. Solid. Struct*., 39(10):2731–2743.

[25] Park. S. K, Gao. X. L, (2006), Bernoulli–Euler beam model based on a modified couple stress theory, *J. Micro. Mech. Micro. Eng*., 16(11):2355–2359.

[26] Kong. S. L, Zhou. S. J, Nie. Z. F, (2008), The size-dependent natural frequency of Bernoulli–Euler micro-beams, *Int. J. Eng. Sci*., 46(5):427–437.

[27] Lam. D. C. C, Yang. F, Chong. A. C. M, (2003), Experiments and theory in strain gradient elasticity, *J. Mech. Phys. Solid*., 51(8):1477–1508.

[28] Kahrobaiyan. M. H, Asghari. M, Rahaeifard. M, Ahmadian. M. T, (2011), A nonlinear strain gradient beam formulation, *Int. J. Eng. Sci*., doi:10.1016/j.ijengsci.2011.01.006.

[29] Kong. S, Zhou. S, Nie. Z, Wang. K, (2009), Static and dynamic analysis of micro beams based on strain gradient elasticity theory, *Int. J. Eng. Sci*., 47:487–498.

[30] Fleck. N. A, Muller. G. M, Ashby. M. F, Hutchinson. J. W, (1994), Strain gradient plasticity: Theory and experiment, *Acta Met. et Mat*., 42(2):475–487.

[31] Stolken. J. S, Evans. A. G, (1998), Microbend test method for measuring the plasticity length scale, *Acta Mat*., 46(14):5109–5115.

[32] Narendar. S, (2011), Nonlocal torsional vibration of nanorods, *J. Nanosci. Nanoeng. Appl*., 1(2):36-51.

[33] Aydogdu. M, (2009), Axial vibration of the nanorods with the nonlocal continuum rod model, *Physica E*, 41: 861–864.

[34] Narendar. S, Gopalakrishnan. S, (2010), Ultrasonic wave characteristics of nanorods via nonlocal strain gradient models, *J. Appl. Phys. (AIP)*, 107:084312-1:8.

[35] Narendar. S, (2011), Terahertz wave propagation in uniform nanorods: a nonlocal continuum mechanics formulation including the effect of lateral inertia, *Physica E*, 43:1015–1020.

[36] Narendar. S, Roy Mahapatra. D, Gopalakrishnan. S, (2011), Prediction of nonlocal scaling parameter for armchair and zigzag single-walled carbon nanotubes based on molecular structural mechanics, nonlocal elasticity and wave propagation, *Int. J. Eng. Sci.*, 49:509-522.

[37] Narendar. S, Gopalakrishnan. S, (2011), Spectral finite element formulation for nanorods via nonlocal continuum mechanics, *J. Appl. Mech*., (Transactions of ASME), 78:061018-1:9.

[38] Narendar. S, Roy Mahapatra. D, Gopalakrishnan. S, (2011), Ultrasonic wave characteristics of a monolayer graphene on silicon substrate, *Comp. Struct.*, 93:1997-2009.

[39] Narendar. S, Gopalakrishnan. S, (2011), Axial wave propagation in coupled nanorod system with nonlocal small scale effects, Comp.: *Part B – Eng.,* 42:2013-2023.

[40] Narendar. S, (2011), Buckling analysis of micro-/nano-scale plates based on two variable refined plate theory incorporating nonlocal small scale effects, *Comp. Struct.,* 93:3093-3103.

[41] Narendar. S, Gopalakrishnan. S, (2011), Nonlocal wave propagation in rotating nanotube, Result. *Phys., 1*: 17-25.

[42] Narendar. S, Gopalakrishnan. S, (2011), Critical buckling temperature of single-walled carbon nanotubes embedded in a one-parameter elastic medium based on nonlocal continuum mechanics, *Physica E*, 43:1185-1191.

[43] Narendar. S, (2011), Mathematical modelling of rotating single-walled carbon nanotubes used in nanoscale rotational actuators, *Def. Sci. J. (Special Issue on Aerospace Avionics and Allied Technologies)*, 61(4):317-324.

[44] Narendar. S, Gopalakrishnan. S, (2011), A revisit to capture the entire behavior of ultrasonic wave dispersion characteristics of single walled carbon nanotubes based on nonlocal elasticity theory and Flügge shell model, *J. Comput. Theor. Nanosci*., (American Scientific Publishers), 8:1933–1944.

[45] Chong. C. M, (2002), Experimental investigation and modeling of size effect in elasticity, Ph.D.,

Hong Kong University of Science and Technology (People's Republic of China), Hong Kong.

[46] Dym. C. L, Shames. I. H, (1984), Solid Mechanics: *A Variational Approach, China Railway Publishing House*, Beijing.

[47] Gheshlaghi. B, Hasheminejad. S. M, Abbasion. S, (2010), Size dependent torsional vibration of nanotubes, *Physica E*, 43:45-48.

[48] Rao. S. S, (2007), Vibration of continuous systems. Hoboken, *New Jersey: John Wiley & Sons, Inc.*