

Vibration analysis of a timoshenko non-uniform nanobeam based on nonlocal theory: An analytical solution

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Abstract

In this article free vibration of a timoshenko nanobeam with variable cross-section is investigated using nonlocal elasticity theory within the scope of continuum mechanics. Small scale effects are modelled after Eringen's nonlocal elasticity theory while the non-uniformity is presented by exponentially varying width through the beam length with constant thickness. Analytical solution is achieved for both Timoshenko beams and nanobeams with different boundary conditions including both ends being simply-supported (S-S), both ends being clamped (C-C) and one end clamped other free (C-F). It is shown that section variation accompanying small scale effects has a noticeable effect on natural frequencies of non-uniform Timoshenko beams at nano-scale. In order to illustrate these effects, Natural frequencies of single-layered graphene nanoribbons (GNRs) with various boundary conditions are obtained for different nonlocal and non-uniform parameter which shows a great sensitivity to non-uniformity in different shape modes.

Keywords: Analytical solution; Free vibration; Nanobeam; Nonlocal; Timoshenko beam; Variable cross-section.

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INTRODUCTION

Nanobeams are one of the most important nano-dimension structures which have attracted a great deal of attention due to the extensive usage in recent years. Lots of researches have been done in order to understand nanomaterials mechanical, chemical, electrical, optical and electronic properties. Nanobeams are used in designing atomic force microscope [1-3], nanowires [4-6], nanoactuators [7-8] and nanoprobes [9-11]. In order to make NEMS devices more efficient, it's necessary to use nanobeams in a more optimum way which having a non-uniform cross-section is one of them. Beams with geometry properties varying along the length with the ability to reduce weight or volume as well as to increase strength and stability of structures have engrossed great deal of attention in engineering designs. Understanding mechanical behaviors of nanostructures in both static problems such as

bending [12-17] and buckling [18-22] and dynamic [23-29] analysis is the key step for designing more efficient NEMS devices. There is many experimental and theoretical studies reported the behavior of nanobeams. Although experimental studies give more accurate results but conducting experiments at nanoscale size is quite expensive and difficult which shows the importance of developing appropriate mathematical models. Recently, there has been a great attention in investigating non-uniform nanobeams behaviors. Pandeya and Singhb [30] studied free vibration of a nanocantilever beam with non-uniform cross-section using finite element methods. The Euler-Bernoulli beam model and Eringen's nonlocal theory were used to model the nanocantilever nanobeam. It was shown that with the introduction of nonlocal effects the frequency of vibration increases. Murmu and Pradhan [31] used differential quadrature (DQ) method to

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investigate the small-scale effect on the vibration of non-uniform nanobeams based on nonlocal elasticity theory which was only provided for cantilever Euler-Bernoulli nanobeams. The study showed that the nonlocal frequency solutions of nanocantilever are larger compared to the classical (local) solutions till a critical height ratio (CHR). Beyond the CHR, nonlocal solutions are lower than the classical (local) solutions. Malekzadeh and Shojaee [32] studied the surface and nonlocal effects on the nonlinear flexural free vibrations of elastically supported non-uniform cross-section nanobeams. Green's strain tensor together with von Kármán assumptions were employed to model the geometrical nonlinearity and DQ method with direct iterative method was adopted to obtain the nonlinear vibration frequencies of nanobeams subjected to different boundary conditions. It was shown that varying the width taper ratio, nonlocal parameter, surface elasticity and initial stress changes the frequency parameter of tapered beams. Hosseini Hashemi and Bakhshi Khaniki [33] studied Bending vibrations of non-uniform Euler beam using the Eringen's nonlocal elasticity theory. Analytical solution was presented for Euler beam theory which the rotating effects were neglected and parametric study was presented. They also studied the effects of this type of non-uniformity on functionally graded beams [34]. Lee and Chang [35] obtained the natural frequency of a non-uniform nanocantilever beam with consideration of surface effects using the nonlocal elastic theory. Chakraverty and Behera [36] presented free vibration of non-uniform Euler-Bernoulli nanobeams based on nonlocal elasticity theory using boundary characteristic orthogonal polynomials implemented in the Rayleigh-Ritz method. Chang [37] studied non-uniform and non-homogeneous nanorods using the theory of nonlocal elasticity. The non-uniform and non-homogeneous nanorod was assumed as hollow with constant thickness. Both clamped and clamped-free boundary conditions were used to model the nanorod. Numerical results were presented using DQ method for first three modes of vibration in non-uniform and non-homogeneous nanorods. It was concluded that the nonlocal frequency is less than the local frequency due to the effect of small length scale. Şimşek [38] studied the free vibration of functionally graded tapered nanorods. Nonlocal elasticity theory was used to model the small scale effects while the material variation was modeled using power law function. Clamped-clamped and clamped-free boundary conditions were considered for nanorods. Free

vibration frequencies were obtained using Galerkin method and the effects of nonlocal parameter, different material composition, taper ratio, different change of the cross-sectional area and the boundary conditions on the free vibration characteristics of non-homogeneous non-uniform nanorods were discussed. Ece *et al.* [39] studied the free vibration in non-uniform isotropic thin beams. Non-uniformity was presented by exponentially varying width with constant thickness. Analytical solution was presented for different boundary conditions and first, second and fifth natural frequency was calculated. It was seen that the non-uniformity in the cross-section has a great influence on the natural frequencies and the mode shapes. Akgoz *et al.* [40] studied the buckling in tapered columns in microscales using modified strain gradient elasticity and Euler-Bernoulli beam theory. Cantilever boundary condition and Rayleigh-Ritz solution method were used to achieve the critical buckling loads in this type of non-uniform microbeams. It was shown that critical buckling loads achieved by modified stress theory are different from those achieved by classical theories which show that classical theories are unable to predict the behavior of microbeams accurately. Zeighampour and Beni [41] studied the free vibration of non-uniform functionally graded nanobeams using strain gradient theory. Euler-Bernoulli beam theory was used to model the nanobeam and it was assumed that nanobeam is resting on a visco-Pasternak medium and material variation happens in longitudinal direction of nanobeam. Linear and nonlinear non-uniformities were discussed and numerical results were presented using differential quadrature method. Results shown that variation of Young's modulus, density, diameter of the nanobeam have a great influence on the natural frequency of non-uniform functionally graded nanobeams.

The dynamic characteristics of a non-uniform Timoshenko nanobeam has far less been studied mainly due to the complexity of analytical solution. In the present work, analytical solution for free vibration analysis of a non-uniform Timoshenko nanobeam is investigated for various support conditions using Eringen's nonlocal elasticity theory. It's assumed that the thickness remains constant while the width varies exponentially along the beam. Nonlocality, non-uniformity, rotary inertia and shear deformation effects on the natural frequency of the beam is studied. Parametric study for non-uniform graphene nanoribbon is done and a schematic view of it is presented in Fig. 1a.

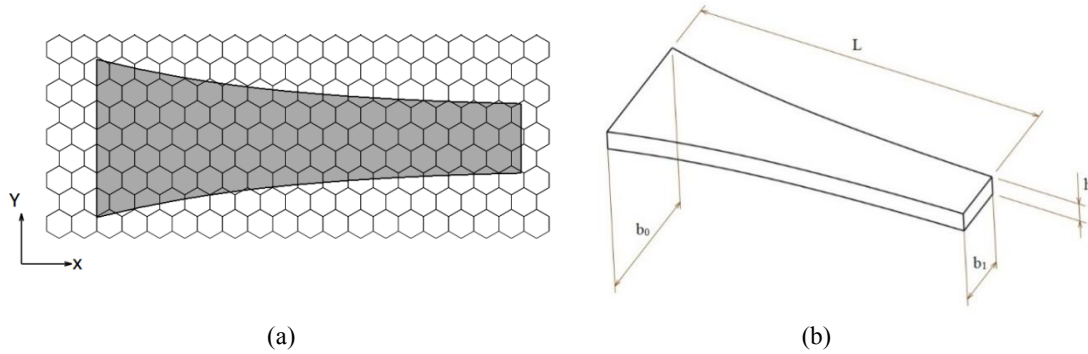


Fig. 1: Schematic view of non-uniform nanobeam (a) Model of single layered graphene nanobeam with exponentially width variation (b) continuum beam model of a single-layered nanobeam.

PROBLEM FORMULATION

Timoshenko beam

The strain-displacement relations Based on the Timoshenko beam theory are given by

$$\epsilon_{xx} = z \frac{d\varphi}{dx} \tag{1}$$

$$\gamma_{xz} = \varphi + \frac{dw}{dx} \tag{2}$$

Where x is the longitudinal coordinate measured from the left end of the beam, z the transverse coordinate measured from the midplane of the beam and ϵ_{xx} is the normal strain, φ is the rotation parameter due to the bending, γ_{xz} the transverse shear stress. The strain energy U is given by

$$U = \frac{1}{2} \int_0^L \int_A (\sigma_{xx} \epsilon_{xx} + \sigma_{xz} \gamma_{xz}) dA dx \tag{3}$$

In which A and L are the cross-sectional area and length of the beam, σ_{xx} and σ_{xz} are the normal and shear stresses. By substituting Eq. (1) and Eq. (2) into Eq. (3), the strain energy may be expressed as

$$U = \frac{1}{2} \int_0^L \int_A \left(\sigma_{xx} z \frac{d\varphi}{dx} + \sigma_{xz} \left(\varphi + \frac{dw}{dx} \right) \right) dA dx \tag{4}$$

$$= \frac{1}{2} \int_0^L \left(M \frac{d\varphi}{dx} + Q \left(\varphi + \frac{dw}{dx} \right) \right) dx$$

Where M is the bending moment and Q is the shear force defined as

$$M = \int_A z \sigma_{xx} dA \tag{5}$$

$$Q = \int_A \sigma_{xz} dA \tag{6}$$

Also the kinetic energy T is given by

$$T = \frac{1}{2} \int_0^L \left\{ \int_A \rho \left(\frac{\partial w}{\partial t} \right)^2 dA + \int_I \rho \left(\frac{\partial \varphi}{\partial t} \right)^2 dl \right\} dx \tag{7}$$

Where ρ is the mass density of the beam. By assuming free harmonic motion Eq. (7) may be expressed as

$$T = \frac{1}{2} \rho \omega^2 \int_0^L \left\{ \int_A W^2 dA + \int_I \varphi^2 dl \right\} dx \tag{8}$$

Where, ω is the circular frequency of vibration and I is the second moment of area. Governing equation of motion is achieved using Lagrange-Hamilton method is presented as

$$\delta(T_{max} - U_{max}) = 0$$

$$= \int_0^L \left(\int_A \rho \omega^2 W \delta W dA + \int_I \rho \omega^2 \varphi \delta \varphi dl - M \frac{d\delta W}{dx} - Q \left(\delta \varphi + \frac{d\delta W}{dx} \right) \right) dx \tag{9}$$

With integrating by parts, we have

$$\int_0^L \left[\left(\rho I \omega^2 \varphi + \frac{dM}{dx} - Q \right) \delta \varphi + \left(\frac{dQ}{dx} + \rho A \omega^2 W \right) \delta W \right] dx - [M \delta \varphi]_0^L - [Q \delta W]_0^L = 0 \tag{10}$$

Since δW and $\delta \varphi$ is arbitrary in $0 < x < L$, the governing equations of motion are obtained as

$$\begin{cases} \frac{dM}{dx} = Q - \rho I \omega^2 \varphi \\ \frac{dQ}{dx} = -\rho A \omega^2 W \end{cases} \tag{11}$$

Nonlocal elasticity theory

Classical elasticity theories don't conflict the atomic theory of lattice dynamics and experimental observation of phonon dispersion by defining the stress at a reference point x in an elastic continuum depends only on the strain at that point. Eringen's nonlocal elasticity involves spatial integrals which represent weighted averages of the contributions of strain tensors of all points in the body to the stress tensor at the given point. Basic equations for a linear homogenous nonlocal elastic body without the body forces are given as

$$\begin{aligned} \sigma_{j,j} &= 0 \\ \sigma_{ij}(x) &= \int \tau(|x-x'|, \alpha) C_{ijkl} \varepsilon_{kl}(x') dV(x'), \quad \forall x \in V \\ \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \end{aligned} \quad (12)$$

Where σ_{ij} is the stress tensor, C_{ijkl} is the fourth-order elasticity tensor, $|x-x'|$ is the distance in Euclidean form and $\tau(|x-x'|, \alpha)$ is the nonlocal modulus or attenuation function incorporating into constitutive equations the nonlocal effects at the reference point x produced by local strain at the source x' . a is the material constant which is defined as $(e_0 a/l)$ depends on the internal (e.g. lattice parameter, granular distance, distance between C-C bonds) and external (e.g. crack length, wavelength) lengths. Due to the difficulty of solving the integral constitutive Eq. (12) can be simplified to equation of differential form as

$$(1 - \alpha^2 l^2 \nabla^2) \sigma = t \quad (13)$$

For a one dimensional elastic material, the Eq. (13) can be written as

$$\begin{cases} \left(1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right) \sigma_{xx}(x) = E \varepsilon_{xx}(x) \\ \sigma_{xz} = G \gamma_{xz} \end{cases} \quad (14)$$

Where $(e_0 a)$ is the scale coefficient which leads to small scale effect and E is the Young's modulus of the nanobeam. Multiplying Eq. (14) by zdA and integrating the result over the area A leads to

$$\begin{cases} M - (e_0 a)^2 \frac{d^2 M}{dx^2} = EI \frac{d\varphi}{dx} \\ Q = K_s GA \left(\varphi + \frac{dw}{dx}\right) \end{cases} \quad (15)$$

Where I is the second moment of area, K_s is the shear correction factor and G is the shear modulus defined using Poisson's ratio ν and Young's modulus as $E/2(1+\nu)$. By substituting Eq. (15) into equation (11), we have

$$M_{nonlocal} = EI(x) \frac{d\varphi}{dx} - (e_0 a)^2 \left[\rho A(x) \omega^2 W + \rho I(x) \omega^2 \frac{d\varphi}{dx} + \rho \frac{dI(x)}{dx} \omega^2 \varphi \right] \quad (16)$$

In this study, cross-section of nanobeam is assumed to vary along the beam. The characteristic height of the cross-section or the thickness of the beam is kept constant and the characteristic width of the cross-section is assumed to vary exponentially along the length of the beam as

$$\begin{cases} b(x) = b_0 e^{Nx} \\ b_1 = b_0 e^{Nl} \end{cases} \quad (17)$$

$$\begin{cases} A(x) = b_0 h e^{Nx} \\ A_0 = b_0 h \\ I(x) = \frac{1}{12} b_0 h^3 e^{Nx} \\ I_0 = \frac{1}{12} b_0 h^3 \end{cases} \quad (18)$$

Where N is the non-uniformity parameter, b_0 and b_1 are the width of the beam at the left and right end of the beam shown in Fig. 1b, I_0 and A_0 are second moment of area and cross-section of the beam at the left end and h is the thickness of the beam. Substituting Eq. (16), Eq. (17) and Eq. (18) into Eq. (11), the equations of motion of non-uniform elastic isotropic nanobeam can be derived as

$$\begin{aligned} EI(x) \frac{d^2 \varphi}{dx^2} + E \frac{dI(x)}{dx} \frac{d\varphi}{dx} - K_s GA \left(\varphi + \frac{dw}{dx}\right) + \rho I \omega^2 \varphi \\ - (e_0 a)^2 \left[\rho A(x) \omega^2 \frac{dW}{dx} + \rho \frac{dA(x)}{dx} \omega^2 W + \rho \frac{dI(x)}{dx} \omega^2 \frac{d\varphi}{dx} + \right. \\ \left. \rho I(x) \omega^2 \frac{d^2 \varphi}{dx^2} + \rho \frac{d^2 I(x)}{dx^2} \omega^2 \varphi + \rho \frac{dI(x)}{dx} \omega^2 \frac{d\varphi}{dx} \right] = 0 \end{aligned} \quad (19)$$

$$K_s G \frac{dA}{dx} \left(\varphi + \frac{dw}{dx}\right) + K_s GA \left(\frac{d\varphi}{dx} + \frac{d^2 w}{dx^2}\right) + \rho A \omega^2 W = 0 \quad (20)$$

Note that with $\alpha=0$, Eq. (19) and Eq. (20) reduces to classical equation of motion of non-uniform elastic Timoshenko beam and with $N=0$, Eq. (19) and Eq. (20) reduces to equation of motion of uniform elastic Timoshenko nanobeam and by having both $\alpha=0$ and $N=0$ Eq. (19) and Eq. (20) reduces to classical equation of motion of uniform elastic Timoshenko beam. Non-dimensional parameters are defined as

$$\begin{aligned} X = \frac{x}{L}, W = \frac{w}{L}, \alpha = \frac{e_0 a}{L}, \lambda^2 = \frac{\rho A \omega^2 L^4}{EI}, \\ X = \frac{x}{L}, W = \frac{w}{L}, \alpha = \frac{e_0 a}{L}, \lambda^2 = \frac{\rho A \omega^2 L^4}{EI}, \end{aligned} \quad (21)$$

Where α denotes the dimensionless nonlocal parameter, λ is the natural frequency parameter, X and W are the dimensionless coordinate measured from the left end of the beam along the length and the dimensionless transverse displacement, η is the dimensionless non-uniformity parameter and ξ is the slenderness ratio. Using these parameters, non-dimensional form of the equations and formulation procedures will be defined as

$$\begin{aligned} \varphi'' \left(\Omega - \frac{\Omega \alpha^2 \lambda^2}{\xi^2} \right) + \varphi' \left(\eta \Omega - 2 \frac{\eta \Omega \alpha^2 \lambda^2}{\xi^2} \right) + \varphi \left(-1 + \frac{\Omega \lambda^2}{\xi^2} \right) - \\ W' (1 + \Omega \alpha^2 \lambda^2) - W (\eta \Omega \alpha^2 \lambda^2) = 0 \end{aligned} \quad (22)$$

$$\varphi' + \eta\varphi + W'' + \eta W' + \Omega\lambda^2 W = 0 \quad (23)$$

By uncoupling Eq. (22) and Eq. (23), two fourth order differential equations will appear:

$$w^{(4)} + C_1 w''' + C_2 w'' + C_3 w' + C_4 w = 0 \quad (24)$$

$$\varphi^{(4)} + C_1 \varphi''' + C_2 \varphi'' + C_3 \varphi' + C_4 \varphi = 0 \quad (25)$$

Where C_1, C_2, C_3 and C_4 are defined as

$$\begin{cases} C_1 = \frac{-3\alpha^2\lambda^2 + 2\zeta^2}{-\alpha^2\lambda^2 + \zeta^2} \\ C_2 = \frac{-\Omega\alpha^2\lambda^4 + \zeta^2\alpha^2\lambda^2 - 2\alpha^2\eta^2\lambda^2 + \Omega\zeta^2\lambda^2 + \zeta^2\eta^2 + \lambda^2}{-\alpha^2\lambda^2 + \zeta^2} \\ C_3 = \frac{\lambda^2\eta(-2\Omega\alpha^2\lambda^2 + 2\zeta^2\alpha^2 + \Omega\zeta^2 + 1)}{-\alpha^2\lambda^2 + \zeta^2} \\ C_4 = \frac{\lambda^2(\zeta^2\alpha^2\eta^2 + \Omega\lambda^2 - \zeta^2)}{-\alpha^2\lambda^2 + \zeta^2} \end{cases} \quad (26)$$

In the present study, the ends of the beam are considered to be simply supported (S), clamped (C) or free (F). The boundary conditions associated with both ends being simply supported (SS), both ends being clamped (CC) and left end being clamped while the right end being free (CF) may be written in the same order as

$$\begin{cases} \text{Case I (SS): } W(0)=0, M(0)=0, W(1)=0, M(1)=0 \\ \text{Case II (CC): } W(0)=0, \varphi(0)=0, W(1)=0, \varphi(1)=0 \\ \text{Case III (CF): } W(0)=0, \varphi(0)=0, Q(1)=0, M(1)=0 \end{cases} \quad (27)$$

Solution procedure

Solution of Eq. (24) and Eq. (25) subjected to either boundary conditions given by Eq. (27) can be written in a general form as

$$W(X) = A_{11}e^{\lambda_1 X} + A_{12}e^{\lambda_2 X} + A_{13}e^{\lambda_3 X} + A_{14}e^{\lambda_4 X} \quad (28)$$

$$\varphi(x) = A_{21}e^{\lambda_1 X} + A_{22}e^{\lambda_2 X} + A_{23}e^{\lambda_3 X} + A_{24}e^{\lambda_4 X} \quad (29)$$

Where λ_1 to λ_4 are depended on the frequency parameter and defined as:

$$\begin{cases} \lambda_1 = -\frac{b}{4a} - S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ \lambda_2 = -\frac{b}{4a} - S - \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ \lambda_3 = -\frac{b}{4a} + S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ \lambda_4 = -\frac{b}{4a} + S - \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \end{cases} \quad (30)$$

In which S, a, b, p and q are defined in appendix A. The constants A_{1i} and A_{2i} are depended on each other and could be related as

$$A_{2i} = -\frac{(\lambda_i^2 + \eta\lambda_i + \Omega\lambda^2)}{(\lambda_i + \eta)} A_{1i}, \quad i = 1, 2, 3, 4 \quad (31)$$

Applying boundary conditions in each case leads to an equations for the determining the natural frequencies.

Case I: Both ends of the beam are simply supported (SS).

The non-dimensional boundary conditions for simply-supported Timoshenko beams are given by

$$W(0) = 0 : A_{11} + A_{12} + A_{13} + A_{14} = 0 \quad (32)$$

$$M(0) = 0 : \begin{bmatrix} \lambda_1 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} A_{21} + \begin{bmatrix} \lambda_2 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} A_{22} + \begin{bmatrix} \lambda_3 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} A_{23} + \begin{bmatrix} \lambda_4 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} A_{24} = 0 \quad (33)$$

$$W(1) = 0 : e^{\lambda_1} A_{11} + e^{\lambda_2} A_{12} + e^{\lambda_3} A_{13} + e^{\lambda_4} A_{14} = 0 \quad (34)$$

$$M(1) = 0 : \begin{bmatrix} \lambda_1 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} e^{\lambda_1} A_{21} + \begin{bmatrix} \lambda_2 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} e^{\lambda_2} A_{22} + \begin{bmatrix} \lambda_3 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} e^{\lambda_3} A_{23} + \begin{bmatrix} \lambda_4 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \end{bmatrix} e^{\lambda_4} A_{24} = 0 \quad (35)$$

By substituting Eq. (31) into Eq.'s (32) to (35), the eigenvalue problem for simply supported condition will be defined as

$$\begin{bmatrix} \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \left[\lambda_1 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_1} & \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \left[\lambda_2 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_2} \\ \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \left[\lambda_3 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_3} & \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \left[\lambda_4 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_4} \\ \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \left[\lambda_3 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_1} & \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \left[\lambda_4 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_2} \\ \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \left[\lambda_1 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_3} & \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \left[\lambda_2 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right] e^{\lambda_4} \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{Bmatrix} = 0 \quad (36)$$

And the frequency parameter is observed by computing eigenvalues of coefficient matrix.

Case II: Both ends of the beam are clamped (CC). The non-dimensional boundary conditions for

clamped Timoshenko beams are given by

$$W(0) = 0 : A_{11} + A_{12} + A_{13} + A_{14} = 0 \quad (37)$$

$$\varphi(0) = 0 : A_{21} + A_{22} + A_{23} + A_{24} = 0 \quad (38)$$

$$W(1) = 0 : e^{\lambda_1} A_{11} + e^{\lambda_2} A_{12} + e^{\lambda_3} A_{13} + e^{\lambda_4} A_{14} = 0 \quad (39)$$

$$\varphi(1) = 0 : e^{\lambda_1} A_{21} + e^{\lambda_2} A_{22} + e^{\lambda_3} A_{23} + e^{\lambda_4} A_{24} = 0 \quad (40)$$

By substituting Eq. (31) into Eq.'s (37) to (40), the eigenvalue problem for clamped condition will be defined as

$$\begin{bmatrix} 1 & 1 \\ \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} & \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \\ e^{\lambda_1} & e^{\lambda_2} \\ \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} e^{\lambda_1} & \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} e^{\lambda_2} \\ 1 & 1 \\ \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} & \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \\ e^{\lambda_3} & e^{\lambda_4} \\ \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} e^{\lambda_3} & \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} e^{\lambda_4} \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{24} \end{Bmatrix} = 0 \quad (41)$$

The frequency parameter is observed by computing eigenvalues of coefficient matrix.

Case III: The left end of the beam is clamped while the right end is free (CF).

The non-dimensional boundary conditions for cantilever Timoshenko beams are given by

$$W(0) = 0 : A_{11} + A_{12} + A_{13} + A_{14} = 0 \quad (42)$$

$$\varphi(0) = 0 : A_{21} + A_{22} + A_{23} + A_{24} = 0 \quad (43)$$

$$Q(1) = 0 : \left(\frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \left(\lambda_1 - \frac{(\eta\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{(\xi^2\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_1} A_{11} + \left(\frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \left(\lambda_2 - \frac{(\eta\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{(\xi^2\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_2} A_{12} + \left(\frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \left(\lambda_3 - \frac{(\eta\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{(\xi^2\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_3} A_{13} + \left(\frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \left(\lambda_4 - \frac{(\eta\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{(\xi^2\alpha^2\lambda^2)}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_4} A_{14} = 0 \quad (44)$$

$$W(1) = 0 : \left(\lambda_1 - \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \right) e^{\lambda_1} A_{11} + \left(\lambda_2 - \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \right) e^{\lambda_2} A_{12} + \left(\lambda_3 - \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \right) e^{\lambda_3} A_{13} + \left(\lambda_4 - \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \right) e^{\lambda_4} A_{14} = 0 \quad (45)$$

The frequency parameter is observed by computing eigenvalues of coefficient matrix by substituting Eq. (31) into Eq.'s (42) to (45) which will be defined as

$$\begin{bmatrix} 1 & 1 \\ \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} & \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \\ CN_1 & CN_2 \\ \left(\lambda_1 - \frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \right) e^{\lambda_1} & \left(\lambda_2 - \frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \right) e^{\lambda_2} \\ 1 & 1 \\ \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} & \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \\ CN_3 & CN_4 \\ \left(\lambda_3 - \frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \right) e^{\lambda_3} & \left(\lambda_4 - \frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \right) e^{\lambda_4} \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{Bmatrix} = 0 \quad (46)$$

Where and are defined in Appendix. By knowing the natural frequencies and the coefficients, the unsteady transverse vibration of the beam can then be written as

$$W = (A_1 e^{\gamma_1 X} + A_2 e^{\gamma_2 X} + A_3 e^{\gamma_3 X} + e^{\gamma_4 X}) [d_1 \cos(\omega t) + d_2 \sin(\omega t)] \quad (47)$$

RESULTS AND DISCUSSION

For different vibrating mode numbers, influence of small length scale and non-uniformity cross-section on the vibration of single layered graphene nanoribbons (SLGNRs) is illustrated. Wang and Wang [42] have shown that the value of $e_0 a$ should be smaller than 2.0 nm for carbon nanotubes also the exact value of nonlocal parameter is not exactly known. The external characteristic length varies so the nonlocal or small scale coefficient parameter $\alpha = e_0 a/l$ is taken from 0 to 0.7 and the non-uniformity parameter $|\eta|$ is also assumed to change from 0 to 1. The analysis presented, describes the nonlocal free vibration of a Timoshenko nanobeam with exponentially varying characteristic width and provides the analytical solutions. The natural frequency for the SS, CC and CF boundary conditions are obtained respectively. According to the symmetric boundary conditions in clamped and simply supported beams, sign of the non-uniformity parameter has no effects on

the results while in cantilever nanobeams the sign of the non-uniformity parameter matters.

In order to verify the validation of present solution procedure, the non-uniformity parameter η is assumed to be zero to compare the present solution with nonlocal Timoshenko beam [42]. In Table 1 the non-dimensional natural frequency parameter of a simply supported beam with various nonlocal parameters are presented while the non-uniformity parameter is assumed to be zero and the results are compared with those obtained by Wang *et al.* [43]. Same works has

been done for different boundary conditions which are presented and compared in Table 2 and Table 3 for clamped and cantilever nanobeams with various nonlocal parameters which show a great agreement between the results.

Influence of the nonlocal parameter on the first 5 modes of vibration is clarified for different kind of boundary conditions which is shown in Fig. 2. The non-uniformity parameter is assumed to be $\eta=1$ and the beams slenderness is $L/D=10$ so the shear deformation and rotary inertia be mentioned due to having a short beam.

Table 1: Natural frequency parameters for a Timoshenko simply supported beam with various scaling effect parameters.

$ \eta $	Mode Number	Natural frequencies (S-S)							
		$\alpha = 0.1$	Wang et al.	$\alpha = 0.3$	Wang et al.	$\alpha = 0.5$	Wang et al.	$\alpha = 0.7$	Wang et al.
0	1	3.02432	3.0243	2.65378	2.6538	2.28668	2.2867	2.01055	2.0106
0	2	5.53036	5.5304	4.20576	4.2058	3.40365	3.4037	2.91589	2.9159
0	3	7.46987	7.4699	5.24441	5.2444	4.16447	4.1644	3.54531	3.5453
0	4	8.98744	8.9874	6.02276	6.0228	4.74356	4.7436	4.02834	4.0283
0	5	10.2061	10.206	6.63336	6.6333	5.20088	5.2009	4.41074	4.4107

Table 2: Natural frequency parameters for a Timoshenko clamped beam with various scaling effect parameters.

$ \eta $	Mode Number	Natural frequencies (C-C)							
		$\alpha = 0.1$	Wang et al.	$\alpha = 0.3$	Wang et al.	$\alpha = 0.5$	Wang et al.	$\alpha = 0.7$	Wang et al.
0	1	4.34712	4.3471	3.78946	3.7895	3.24201	3.2420	2.83829	2.8383
0	2	6.49518	6.4952	4.94276	4.9428	3.99398	3.9940	3.41916	3.4192
0	3	8.19691	8.1969	5.84601	5.8460	4.67685	4.6769	3.99605	3.9961
0	4	9.54474	9.5447	6.47620	6.4762	5.11311	5.1131	4.34549	4.3455
0	5	10.6488	10.649	7.01703	7.0170	5.52828	5.5283	4.69860	4.6986

Table 3: Natural frequency parameters for a Timoshenko cantilever beam with various scaling effect parameters.

$ \eta $	Mode Number	Natural frequencies (C-F)							
		$\alpha = 0.1$	Wang et al.	$\alpha = 0.3$	Wang et al.	$\alpha = 0.5$	Wang et al.	$\alpha = 0.7$	Wang et al.
0	1	1.86496	1.8650	1.89986	1.8999	2.00238	2.0024	-	-
0	2	4.35057	4.3506	3.65941	3.6594	2.89033	2.8903	-	-
0	3	6.60910	6.6091	5.07623	5.0762	-	-	-	-
0	4	8.31508	8.3151	5.78749	5.7875	-	-	-	-
0	5	9.67045	9.6705	6.58342	6.5834	-	-	-	-

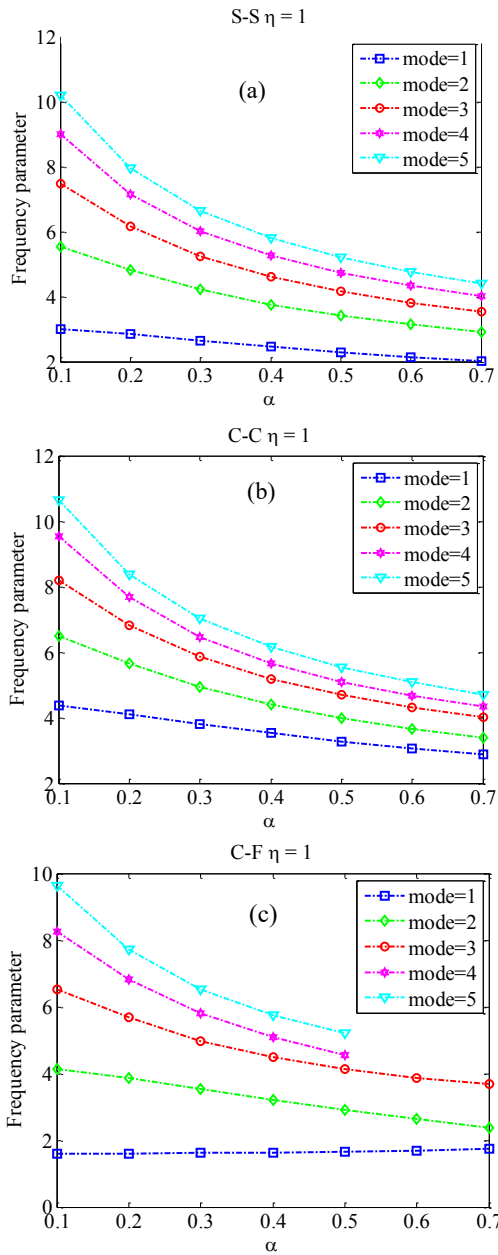


Fig. 2: Variation of frequency parameter of a beam with exponentially varying width ($\eta=1$) and $L/D = 10$ for different nonlocal parameter (a) simply supported (b) Clamped (c) Cantilever.

In Fig. 2a, it can be seen that by increasing the small scale effects in the non-uniform Timoshenko nanobeam with simply supported end, the natural frequency parameter decreases permanently. The same behavior is also seen for the clamped ended non-uniform Timoshenko nanobeams in Fig. 2b. For the cantilever condition, the frequency parameter of the first mode of vibration starts

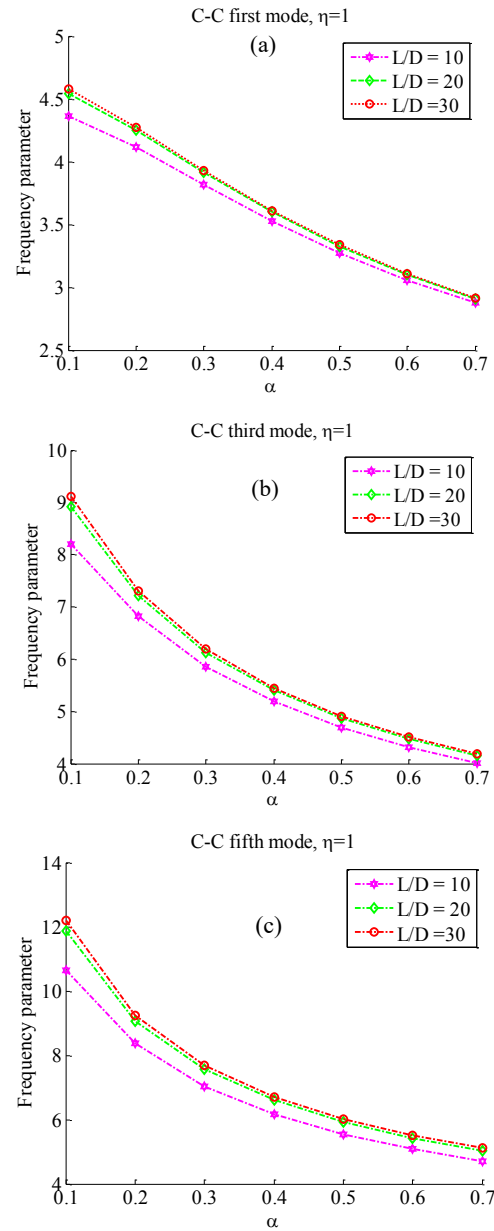


Fig. 3: Rotary inertia and transverse shear deformation effects on frequency parameter for clamped non-uniform beam with different nonlocal parameters and $\eta=1$ (a) simply supported (b) Clamped (c) Cantilever.

increasing by increasing the small scale terms but for the higher modes of vibration, the frequency parameter decreases constantly by increasing the small scale parameter which can be seen in Fig. 2c.

To illustrate the influence of slenderness parameter on the frequency parameter of non-uniform clamped Timoshenko nanobeams for different numbers of nonlocal parameter, the

slenderness parameter assumed to be $L/D = 10, 20$ and 30 while the nonlocal parameter is changed from 0.1 to 0.7 and the results for frequency parameter is presented in Fig. 3a for simply-

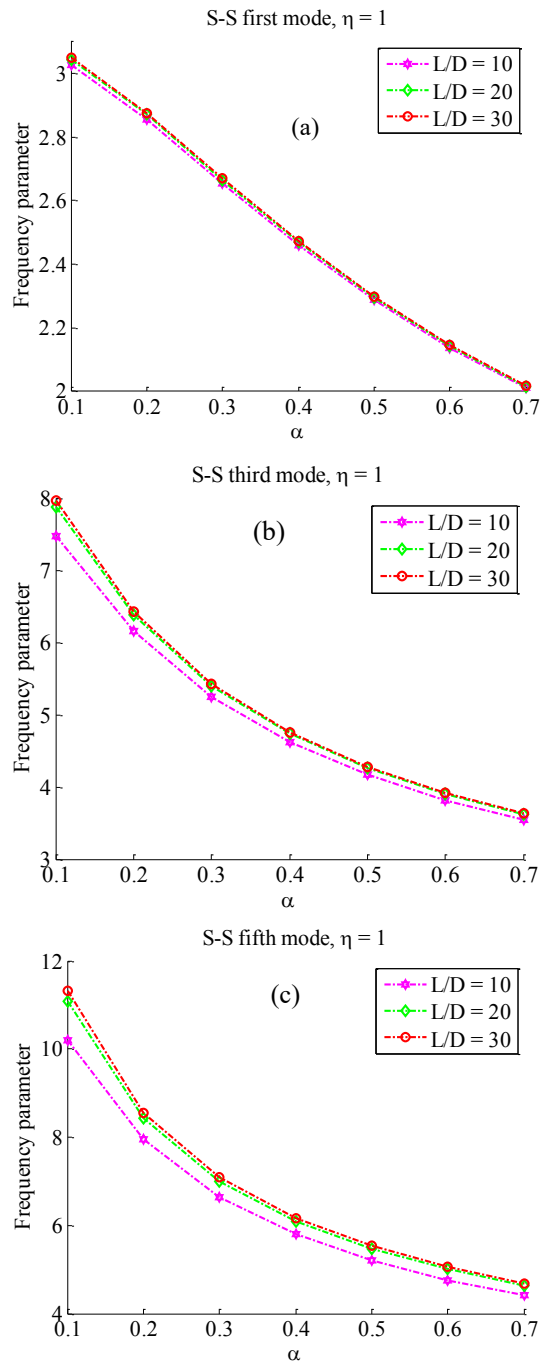


Fig. 4: Rotary inertia and transverse shear deformation effects on frequency parameter for simply supported non-uniform beam with different nonlocal parameters and $\eta=1$ (a) First Mode (b) Third Mode (c) Fifth mode.

supported, Fig. 3b for clamped and Fig. 3c for cantilever nanobeams. As it's shown, for all the boundary conditions presented in this paper, by increasing the slenderness ratio parameter which leads to decreasing the effects of rotary inertia and shear deformation, the frequency parameter increases independent from the nonlocal parameters amount. It is also shown that the frequency parameter shows more sensitivity to the changes in slender ratio parameter in the small number of it and it merges to a specific number for higher slender ratio parameters. It should be noted that by having higher slender ratio parameter, effects of rotary inertia disappears and the results are the same as Euler-Bernoulli nanobeams.

In the same way, in Fig. 4 the influence of slenderness parameter on the frequency parameter of non-uniform simply supported Timoshenko nanobeams for different numbers of nonlocal parameter is studied while the slenderness parameter assumed to be $L/D = 10,$

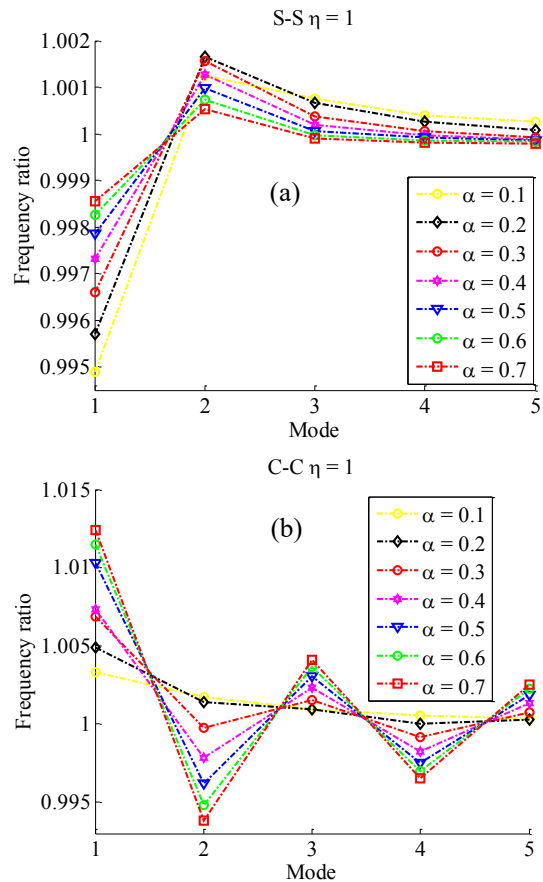


Fig. 5: Small scale effect on the frequency with $L/D = 10$ and $\eta=1$ (a) simply supported (b) Clamped.

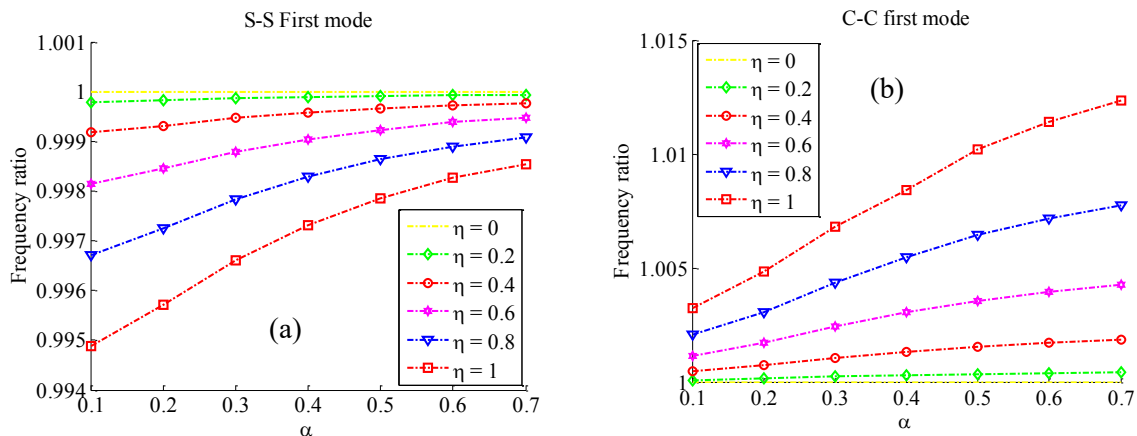


Fig 6. Non-uniformity and nonlocal effect on the frequency for first mode with $L/D = 10$ (a) simply supported (b) Clamped.

20 and 30 and the nonlocal parameter is varied from 0.1 to 0.7 which the results show the same behavior with the clamped ended Timoshenko nanobeam. Results are presented for first mode of vibration in Fig. 4a and for third and fifth mode of vibration in Fig. 4b and Fig. 4c.

Also sensitivity of frequency parameter to the changes in nonlocal parameter for different vibration modes are presented in Fig. 5a for simply supported and Fig. 5b for clamped non-uniform Timoshenko nanobeams while the non-uniformity parameter is assumed to be $\eta=1$. It is shown that increasing the non-uniform parameter has more effect on frequency ratio of lower vibration modes. Variation of frequency parameter by increasing the non-uniformity parameter from 0 to 1 and changing the nonlocal parameter while the value of L/D is set to be 10 is presented for first mode of vibration for simply supported nanobeams in Fig. 6a and clamped nanobeams in Fig. 6b. The sensitivity to changes in nonlocal parameter is more for the non-uniformed Timoshenko nanobeams with higher non-uniformity scale.

CONCLUSION

In this study, general analytical solution based on the Eringen’s nonlocal elasticity theory is formulated for non-uniform timoshenko nanobeams to describe the free vibration of single-layered graphene nanoribbons (GNRs) with variable cross-section in different boundary conditions. Results are achieved and parametric study is done in different manners by varying the nonlocal parameter in order to show the small scale effects, varying the non-uniformity

parameter to obtain the effects of having a non-uniform cross-section and changing value of L/D to understanding the effects of rotary inertia and shear deformation on frequency parameter in different modes. It is shown that:

When the nonlocal effect was taken into account that without consideration non-uniformity, in this manner, the frequency ratio of the beam decreased by increasing the nonlocal parameter except for the first frequency mode of cantilever nanobeam.

For the time when non-uniformity effects were also taken into account without consideration nonlocal effect, the frequency ratio of the beam changes differently with increase in non-uniformity parameter depending on the boundary conditions.

By having both nonlocal and non-uniformity effects in the system, natural frequency changes depending on the boundary condition which is presented in the study.

Increasing the value of L/D in non-uniform Timoshenko nanobeam will merge the answers to those achieved by solving non-uniform Euler nanobeam problem by eliminating the effects of rotary inertia and shear deformation.

Rotary inertia and shear deformation has the less effect on first mode of vibration in non-uniform nanobeam. These effects are much more for higher modes of vibration.

In non-uniform nanobeams, increasing the nonlocal parameter has more effect on first mode of vibration and the sensitivity decreases in higher modes.

Increasing non-uniformity in cantilever nanobeams will lead to having more frequency modes.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

APPENDIX

Appendix A

The expanded form of p, q, Sand Q are

$$\left\{ \begin{aligned} p &= \frac{8ac - 3b^2}{8a^2} \\ q &= \frac{b^3 - 4abc + 8a^2d}{8a^3} \\ S &= \frac{1}{2} \sqrt{-\frac{2}{3}p + \frac{1}{3a} \left(Q + \frac{\Delta_0}{Q} \right)} \\ Q &= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \end{aligned} \right. \quad (A.1)$$

Where a, b, c, d, e, Δ₀ and Δ₁ are defined as

$$\left\{ \begin{aligned} \Delta_0 &= c^2 - 3bd + 12ae \\ \Delta_1 &= 2c^3 - 9bcd + 27b^2e + 27ad^2 - 72ace \\ a &= 1 \\ b &= 2\eta \\ c &= \eta^2 + \alpha^2\lambda^2 \\ d &= 2\alpha^2\eta\lambda^2 \\ e &= \lambda^2(\alpha^2\eta^2 - 1) \end{aligned} \right. \quad (A.2)$$

Appendix B

Parameters CN₁, CN₂, CN₃ and CN₄ are defined as

$$CN_1 = \left(\frac{(\lambda_1^2 + \eta\lambda_1 + \Omega\lambda^2)}{(\lambda_1 + \eta)} \left(\lambda_1 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{\xi^2\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_1} \quad (B.1)$$

$$CN_2 = \left(\frac{(\lambda_2^2 + \eta\lambda_2 + \Omega\lambda^2)}{(\lambda_2 + \eta)} \left(\lambda_2 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{\xi^2\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_2} \quad (B.2)$$

$$CN_3 = \left(\frac{(\lambda_3^2 + \eta\lambda_3 + \Omega\lambda^2)}{(\lambda_3 + \eta)} \left(\lambda_3 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{\xi^2\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_3} \quad (B.3)$$

$$CN_4 = \left(\frac{(\lambda_4^2 + \eta\lambda_4 + \Omega\lambda^2)}{(\lambda_4 + \eta)} \left(\lambda_4 - \frac{\eta\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) + \frac{\xi^2\alpha^2\lambda^2}{(\xi^2 - \alpha^2\lambda^2)} \right) e^{\lambda_4} \quad (B.4)$$

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