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ORIGINAL ARTICLE

Bending, buckling, and vibration analysis of functionally graded nanobeams using an inverse trigonometric beam theory

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Abstract

In this study, an inverse trigonometric nanobeam theory is applied for the bending, buckling, and free vibration analysis of nanobeams using Eringen's nonlocal theory. The present theory satisfies zero shear stress conditions at the top and bottom surfaces of the nanobeam using constitutive relations. Equations of motion are derived by applying Hamilton's principle. The present theory is applied for the analysis of functionally graded material nanobeams. All problems are solved by using the Navier technique. For the comparison purpose, the numerical results are generated by using the third shear deformation theory of Reddy, first-order shear deformation theory of Timoshenko, and classical beam theory of Bernoulli-Euler considering the nanosize effects. The present results are found in good agreement with those of higher order theories.

Keywords: *Bending; Buckling; Vibration; Eringen's Nonlocal Theory; FG Nanobeam; Trigonometric Nanobeam Theory.*

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INTRODUCTION

In recent years, nanobeams are widely used in micro electromechanical systems, micro sensors, micro actuators, etc. hence requires an accurate analysis of the nanobeam considering the influence of the small size on bending, buckling, and vibration behaviour. In general, the classical continuum theories failed to accurately predict these responses of nanobeams. To capture the small size effects, there are various nonlocal continuum theories developed to describe the size-dependent phenomenon [1-3]. Eringen's nonlocal theory, strain gradient theory, couple stress theory, and surface elasticity theory are the available approaches in the literature to capture the small size effects of nanobeams. Among those, Eringen's nonlocal theory [4] is widely used in the literature for nanobeams. It is well known to the research community that in the case of thick beams, the classical beam theory (CBT) [5] and the first-order shear deformation theory (FSDT) [6] are not accurate to capture the nonclassical effects which forced the researchers to develop higher-order beam theories to capture the nonclassical effects of deformations [7-9]. Many researchers have presented bending, buckling, and free vibration analysis of isotropic nanobeams using classical and higher-order nonlocal beam theories [10-25].

Functionally graded (FG) materials are advanced composite materials in which material properties such as Young's modulus, shear modulus, and density are graded through-the-thickness of the beam. FG nanobeams have wide applications in nanotechnology. Simsek and Yurtcu [26] have implemented the FSDT accounting nano-size effect for the bending and buckling analysis of FG nanobeams based on Navier's technique. Simsek and Reddy [27] presented the static, buckling, and

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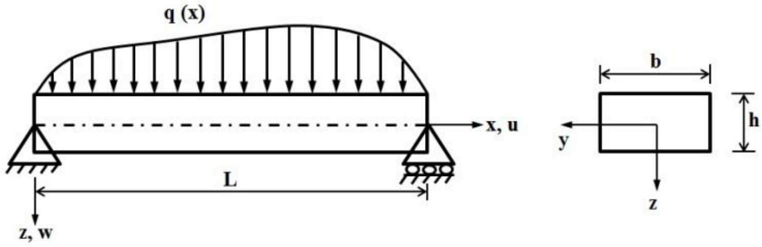

Fig. 1. Geometry and coordinate system of nanobeam.

free vibration analysis of FG nanobeams using different nonlocal theories formulated via unified higher-order beam theory. Akgoz and Civalek [28] and Lei *et al.* [29] applied trigonometric shear deformation theory in conjunction with Navier's method for the bending analysis of FG nanobeams. Ebrahimi and Barati [30] extended the third-order shear deformation theory of Reddy for the free vibration analysis of functionally graded nanobeams using Navier's technique. Ansari *et al.* [31] applied the FSDT for the bending, buckling, and free vibration analysis of FG nanobeams. Yu *et al.* [32] developed a nonlocal beam theory considering normal stretching effects for the static and free vibration analysis of FG nanobeams. Aria and Friswell [33] applied the FSDT along with the finite element technique to investigate the free vibration and buckling behaviour of FG nanobeams. Reddy [34] presented couple-stress theories for FG nanobeams. Eltaher *et al.* [35] have applied the CBT along with the finite element method in conjunction with Eringen's nonlocal theory to study static and buckling responses of FG nanobeams. Khaniki [36] presented a vibration analysis of axially graded FG nanobeams using Eringen's nonlocal theory and modified differential quadrature method. Salamat-talab *et al.* [37] applied the third-order shear deformation theory for the static and dynamic analysis of FG nanobeams using couple stress theory. Zenkour and Ebrahimi [38] have applied the TSDT of Reddy for the buckling analysis of FG nanobeams resting on an elastic foundation based on Eringen's nonlocal theory. Recently Sayyad and Ghugal [39] presented a unified formulation of various nanobeam theories for the bending, buckling, and free vibration of FG nanobeams.

A good number of research papers have been published in the last decade on the bending, buckling, and free vibration analysis of FG nanobeams using classical theories such as the FSDT and the CBT. However, research papers on the bending, buckling, and free vibration analysis of FG nanobeams using higher-order nonlocal beam theories are limited. Therefore, the objective of the present study is to develop and apply a new non-polynomial type higher-order nonlocal beam theory to investigate the bending, buckling, and free vibration responses of FG nanobeams. Hence, the development of a new non-polynomial type higher order nonlocal beam theory and its applications to the bending, buckling, and free vibration analysis of FG nanobeams can be considered as the novelty for this work. An inverse trigonometric function introduced by Nguyen *et al.* [40] is used as a kinematic shape function in this study to develop the present nonlocal beam theory. This function is also used by Sayyad and Ghugal [41] to investigate responses of local FG beams. In the present study, this theory is applied for the bending, buckling, and free vibration analysis of FG nanobeams. The material properties of functionally graded beams are varied through-the-thickness according to the power-law. The theory gives the realistic variation of transverse shear stress through-the-thickness of the beam satisfying boundary conditions at the top and bottom surfaces of the beam. This theory does not require any shear correction factor to account for strain energy due to shear deformation. Equations of motion are derived within the framework of Hamilton's principle. Analytical solutions are obtained using Navier's solution technique. Numerical results are compared with previous literature. The effects of the power-law index and nonlocal parameter on deflections, buckling loads, and fundamental frequencies are investigated.

MATERIALS AND METHODS

A functionally graded (FG) nanobeam as shown in Fig. 1 is considered for the mathematical formulation. A beam has a rectangular cross-

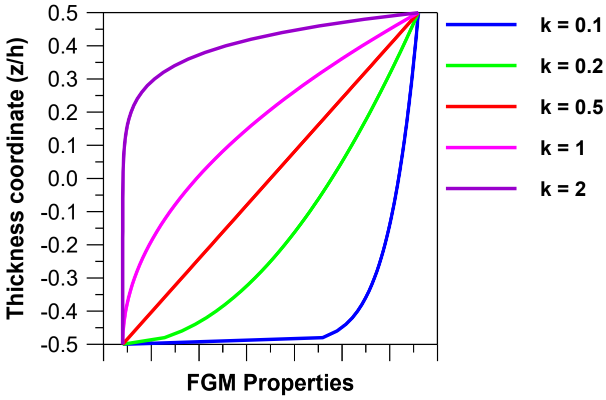

Fig. 2. Variation of material properties for various values of the power-law coefficient.

section with length ***L*** and cross-sectional dimensions ***b*** and ***h***. The beam is subjected to transverse load ***q***(*x*) and axial load $\boldsymbol{N_0}$.

The material properties of FG nanobeam are varying from top to bottom surfaces of the beam using the power-law. The power-law for the material gradation was first introduced by Wakashima *et al.* [42] which is widely used by many researchers for the modeling of functionally graded beams. Modulus of elasticity (*E*) and density (ρ) of the nanobeam are graded through-the-thickness using Eq. (1)

$$E(z)=E_{\mathrm{m}}+(E_{\mathrm{c}}-E_{\mathrm{m}})\left(\frac{1}{2}+\frac{z}{h}\right)^{k}$$
$$\rho(z)=\rho_{\mathrm{m}}+(\rho_{\mathrm{c}}-\rho_{\mathrm{m}})\left(\frac{1}{2}+\frac{z}{h}\right)^{k} \quad (1)$$

Here

$$E(z)=E_{\mathrm{m}} \quad \text{and} \quad \rho(z)=\rho_{\mathrm{m}} \quad \text{at } z=-h/2, k=\infty$$
$$E(z)=E_{c} \quad \text{and} \quad \rho(z)=\rho_{c} \quad \text{at } z=h/2, k=0 \quad (2)$$

where subscripts *m* and *c* are corresponding to metal and ceramic, respectively; *k* represents the power-law coefficient. The range of the value of the power-law coefficient is 0 to ∞. At *k* = 0, the nanobeam is made up of purely ceramic whereas at *k* = ∞ the nanobeam is made up of purely metal. Eq. (2) shows that the top surface of the beam is made up of metal whereas the bottom surface is ceramic. The variation of material properties according to the power-law is shown in Fig. 2. The values of *k* = 0.1, 0.2, 0.5,1, and 2 are taken to plot the variation of material properties according to the power-law. However, one can take any value of the power-law coefficient in the range of 0 to ∞.

The displacement field of the present nonlocal beam theory is developed using the inverse trigonometric shape function to get the traction free boundary conditions on the top and bottom surfaces of the beam.

$$u(x,z,t)=u_0(x,t)-z\frac{\partial w_0}{\partial x}+$$
$$\left[\cot^{-1}\left(\frac{h}{z}\right)-\frac{16}{15}\left(\frac{z^3}{h^3}\right)\right]\phi(x,t) \quad (3)$$
$$w(x,t)=w_0(x,t)$$

where *u* and *w* are the axial (*x*-direction) and transverse (*z*-direction) displacements of any arbitrary point in the beam domain; u_0 and w_0 are the displacements of any arbitrary point on the neutral axis of the beam in the *x*- and *z*-directions, respectively; ϕ is the shear slope. An inverse trigonometric shape function is used in the axial displacement to account for the effects of transverse shear deformation. The nonzero strain quantities associated with the present theory are calculated using the strain-displacement relationships of the linear theory of elasticity.

$$\varepsilon_x=\frac{\partial u}{\partial x}=\varepsilon_x^0+z\,k_x^b+f(z)k_x^s$$
$$\gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=g(z)\gamma_{xz}^0 \quad (4)$$

Where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\quad k_x^b=-\frac{\partial^2 w_0}{\partial x^2},\quad k_x^s=\frac{\partial \phi}{\partial x},\quad \gamma_{xz}^0=\phi,$$
$$f(z)=\cot^{-1}\left(\frac{h}{z}\right)-\frac{16}{15}\left(\frac{z^3}{h^3}\right),\quad g(z)=\frac{h}{h^2+z^2}-\frac{48}{15}\left(\frac{z^2}{h^3}\right) \quad (5)$$

The bending and transverse shear stresses at any arbitrary point in the beam domain are obtained by using the one dimensional Hooke's law.

$$\sigma_x = E(z)\left[\varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s\right]$$
$$\tau_{xz} = \frac{E(z)}{2(1+\mu)}\left[g(z)\gamma_{xz}^0\right] \tag{6}$$

where ***E*(*z*)** is Young's modulus of the FG beam varying through-the-thickness according to Eq. (1). The second of Eqs. (6) satisfies the traction free boundary conditions at the top and bottom surfaces of the beam.

Hamilton's principle in conjunction with the present theory is used to derive equations of motion. Hamilton's principle considering the strain energy (δU), the potential energy (δV), and the kinetic energy (δK) is written as follows:

$$\int_{t_1}^{t_2}\left(\delta U - \delta V + \delta K\right)dt = 0 \tag{7}$$

where t_1 and t_2 are the initial and final time; δ is the variational operator. The final expressions of all energies are as follows

$$\delta U = b\int_0^L\int_{-h/2}^{h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz}\right)dz\,dx =$$
$$\int_0^L\left(N_x\,\delta\varepsilon_x^0 + M_x^b\,\delta k_x^b - M_x^s\,\delta k_x^s + Q\,\delta\gamma_{xz}^0\right)dx \tag{8}$$

$$\delta V = \int_0^L\left(q(x)\delta w + N_0\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right)dx \tag{9}$$

$$\delta K = \int_0^L\int_{-h/2}^{h/2}\rho(z)\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 w}{\partial t^2}\delta w\right)dz\,dx \tag{10}$$

where $q(x)$ and N_0 are the transverse and axial loads, respectively; $\rho(z)$ is the mass density varying through-the-thickness of the beam according to Eq. (1); N_x, M_x^b, M_x^s, Q are the force and moment resultants which are obtained by integrating stresses over the thickness coordinates (z = -h/2 to h/2).

$$\left(N_x, M_x^b, M_x^s\right) = b\int_{-h/2}^{h/2}\left[1, z, f(z)\right]\sigma_x\,dz$$
$$Q = b\int_{-h/2}^{h/2} g(z)\,\tau_{xz}\,dz \tag{11}$$

where

$$N_x = A\varepsilon_x^0 + Bk_x^b + Ck_x^s$$
$$M_x^b = B\varepsilon_x^0 + Dk_x^b + Fk_x^s$$
$$M_x^s = C\varepsilon_x^0 + Fk_x^b + Hk_x^s \tag{12}$$
$$Q = J\gamma_{xz}^0$$

where *A, B, C, D, F, H, J* are the stiffness coefficients defined as follows:

$$\left(A,B,C,D,F,H,J\right) =$$
$$b\int_{-h/2}^{h/2} E(z)\left[1, z, f(z), z^2, zf(z), f^2(z), g^2(z)\right]dz \tag{13}$$

Equations of motion are derived by substituting Eqs. (8)-(10) into Eq. (7) and then integrating by parts. Final equations of motion are obtained by setting the coefficients of unknown variables ($\delta u_0, \delta w_0, \delta\phi$) equal to zero as follows:

$$\delta u_0:\ \frac{\partial N_x}{\partial x} = I_1\frac{\partial^2 u_0}{\partial t^2} - I_2\frac{\partial^3 w_0}{\partial x\partial t^2} + I_3\frac{\partial^2\phi}{\partial t^2}$$
$$\delta w_0:\ \frac{\partial^2 M_x^b}{\partial x^2} = -q + N_0\frac{\partial^2 w_0}{\partial x^2} + I_2\frac{\partial^3 u_0}{\partial x\partial t^2} +$$
$$I_1\frac{\partial^2 w_0}{\partial t^2} - I_4\frac{\partial^4 u_0}{\partial x^2\partial t^2} + I_5\frac{\partial^3\phi}{\partial x\partial t^2} \tag{14}$$
$$\delta\phi:\ \frac{\partial M_x^s}{\partial x} - Q = I_3\frac{\partial^2 u_0}{\partial t^2} - I_5\frac{\partial^3 u_0}{\partial x\partial t^2} + I_6\frac{\partial^2\phi}{\partial t^2}$$

where I_1, I_2, I_3, I_4, I_5, I_6 are the inertia coefficients defined as follows:

$$\left(I_1, I_2, I_3, I_4, I_5, I_6\right) =$$
$$b\int_{-h/2}^{h/2}\rho(z)\left[1, z, f(z), z^2, zf(z), f^2(z)\right]dz \tag{15}$$

The kinematic and the natural boundary conditions at x = 0, L are of the following form

$$\text{Either}\begin{Bmatrix} u_0 = 0 \\ \phi = 0 \\ w_0 = 0 \\ \dfrac{\partial w_0}{\partial x} = 0 \end{Bmatrix}\text{or}$$
$$\begin{Bmatrix} N_x = 0 \\ M_x^s = 0 \\ \dfrac{\partial M_x^b}{\partial x} - N_0\dfrac{\partial w_0}{\partial x} - I_2\dfrac{\partial^2 u_0}{\partial t^2} + I_4\dfrac{\partial^3 w_0}{\partial x\partial t^2} - I_5\dfrac{\partial^2\phi}{\partial t^2} = 0 \\ M_x^b = 0 \end{Bmatrix} \tag{16}$$

Eringen's non-local theory

Eringen's nonlocal theory is used to account for the small-scale effect of nanobeams. According to Eringen's nonlocal theory, nonlocal stresses can be calculated using the following relation.

$$\sigma_{ij}^{NL} = \int_{x_0} k\left(\left|x - x_0\right|, n\right)\sigma_{ij}^{L}\, dx \tag{17}$$

where σ_{ij}^{NL} is the nonlocal stress tensor, σ_{ij}^{L} is the local stress tensor; k is the kernel function, n is the nonlocal parameter calculated using the material constant (e_0), and the internal characteristic length (a) i.e. $n=(e_0 a)^2$; $|x-x_0|$ is the neighborhood distance. Therefore, for functionally graded materials, nonlocal stresses can be obtained using the following constitutive relations.

$$\begin{aligned}
&\sigma_x - n\frac{\partial^2 \sigma_x}{\partial x^2} = E(z)\varepsilon_x \\
&\tau_{xz} - n\frac{\partial^2 \tau_{xz}}{\partial x^2} = \frac{E(z)}{2(1+\mu)}\gamma_{xz}
\end{aligned} \tag{18}$$

The local stresses for the FG beams are recovered by setting a nonlocal parameter equals to zero ($n=0$). Using Eq. (18), the stress resultants presented in Eq. (12) are obtained as follows:

$$\begin{aligned}
&N_x - n\frac{\partial^2 N_x}{\partial x^2} = A\varepsilon_x^0 + Bk_x^b + Ck_x^s \\
&M_x^b - n\frac{\partial^2 M_x^b}{\partial x^2} = B\varepsilon_x^0 + Dk_x^b + Fk_x^s \\
&M_x^s - n\frac{\partial^2 M_x^s}{\partial x^2} = C\varepsilon_x^0 + Fk_x^b + Hk_x^s \\
&Q - n\frac{\partial^2 Q}{\partial x^2} = J\gamma_{xz}^0
\end{aligned} \tag{19}$$

Finally, the nonlocal equations of motion in terms of unknown displacement variables (u_0, w_0, ϕ) are derived by substituting Eq. (19) into Eq. (14).

$$\begin{aligned}
&\delta u_0:\quad A\frac{\partial^2 u_0}{\partial x^2} - B\frac{\partial^3 w_0}{\partial x^3} + C\frac{\partial^2 \phi}{\partial x^2} = I_1\frac{\partial^2 u_0}{\partial t^2} - nI_1\frac{\partial^4 u_0}{\partial x^2 \partial t^2} \\
&\delta w_0:\quad B\frac{\partial^3 u_0}{\partial x^3} - D\frac{\partial^4 w_0}{\partial x^4} + F\frac{\partial^3 \phi}{\partial x^3} = I_1\frac{\partial^2 w_0}{\partial t^2} - I_4\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \\
&I_5\frac{\partial^3 \phi}{\partial x \partial t^2} - q(x) + N_0\frac{\partial^2 w_0}{\partial x^2} - nN_0\frac{\partial^4 w_0}{\partial x^4} \\
&-nI_1\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + nI_4\frac{\partial^6 w_0}{\partial x^4 \partial t^2} - nI_5\frac{\partial^5 \phi}{\partial x^3 \partial t^2} + n\frac{\partial^2 q(x)}{\partial x^2} \\
&\delta\phi:\quad C\frac{\partial^2 u_0}{\partial x^2} - F\frac{\partial^3 w_0}{\partial x^3} + H\frac{\partial^2 \phi}{\partial x^2} - J\phi = \\
&I_6\frac{\partial^2 \phi}{\partial t^2} - I_5\frac{\partial^3 w_0}{\partial x \partial t^2} - nI_6\frac{\partial^4 \phi}{\partial x^2 \partial t^2} + nI_5\frac{\partial^5 w_0}{\partial x^3 \partial t^2}
\end{aligned} \tag{20}$$

Eq. (20) represents the set of equations of motion of FG nanobeam. Equations of motion of a local beam can be recovered from these equations by setting the nonlocal parameter equal to zero ($n=0$). The maximum value of n can be infinity. As the value of the nonlocal parameter increases, the size of the nanobeam decreases.

In this study, the Navier technique is used to obtain analytical solutions for the bending, buckling, and free vibration analysis of simply supported FG nanobeam. The beam has following boundary conditions at $x = 0$ and $x = L$.

$$u_0 = w = M_x^b = M_x^s = 0 \tag{21}$$

To satisfy the above boundary conditions, the unknown variables are assumed in the following form of the Fourier series.

$$\begin{Bmatrix} u_0 \\ w_0 \\ \phi \end{Bmatrix} = \sum_{m=1,3,5}^{\infty} \begin{Bmatrix} u_m \cos\alpha x \\ w_m \sin\alpha x \\ \phi_m \cos\alpha x \end{Bmatrix} e^{i\omega t} \tag{22}$$

where $i=\sqrt{-1}$, $\alpha = m\pi / L$, (u_m, w_m, ϕ_m) are unknown coefficients, and ω is the natural frequency. The transverse uniform load $q(x)$ is also expanded in the Fourier sine series as

$$q(x) = \sum_{m=1,3,5}^{\infty} \frac{4q_0}{m\pi}\sin\alpha x \tag{23}$$

where q_0 is the maximum intensity of the uniform load. Solutions for the bending, buckling, and free vibration problems are obtained by substituting Eqs. (22) and (23) into the Eq. (20).

$$[K]\{\Delta\} = \{F\} \tag{24}$$

$$\{[K] - N_0[N]\}\{\Delta\} = 0 \tag{25}$$

$$\{[K] - \omega^2[M]\}\{\Delta\} = 0 \tag{26}$$

One can note that for bending (N_0 = 0) and buckling problems, time-dependent terms are neglected from the equations of motion whereas for the buckling and free vibration problems transverse load $q(x)$ is taken as zero. The elements of the stiffness matrix [K], mass matrix [M], geometric matrix [N], and the displacement vector $\{\Delta\}$ are as follows.

$$[K]=\begin{bmatrix} A\alpha^2 & -B\alpha^3 & C\alpha^2 \\ -B\alpha^3 & D\alpha^4 & -F\alpha^3 \\ C\alpha^2 & -F\alpha^3 & H\alpha^2+J \end{bmatrix},$$

$$[N]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(\alpha^2+n\alpha^4\right) & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \{\Delta\}=\begin{Bmatrix} u_m \\ w_m \\ \phi_m \end{Bmatrix}, \tag{27}$$

$$\{F\}=\frac{4q_0}{m\pi}\begin{Bmatrix} 0 \\ 1+n\alpha^2 \\ 0 \end{Bmatrix},$$

$$[M]=\begin{bmatrix} I_1(1+n\alpha^2) & 0 & 0 \\ 0 & I_1(1+n\alpha^2)+I_4\alpha^2\left(1+n\alpha^2\right) & -I_5\alpha(1+n\alpha^2) \\ 0 & -I_5\alpha(1+n\alpha^2) & I_6(1+n\alpha^2) \end{bmatrix}$$

RESULTS AND DISCUSSION

Solutions of Eqs. (24) through (26) give transverse deflections, critical buckling loads, and natural frequencies of simply supported functionally graded nanobeams. The numerical results obtained using the present theory are compared with existing literature. The following non-dimensional forms and material properties are used.

Isotropic nanobeam

E = 1 GPa and μ= 0.3

$$\bar{w}=\frac{100EI}{q_0L^4}w,\quad \bar{N}=\frac{12L^2}{EI}N_0,\quad \bar{\omega}=\omega L^2\sqrt{\frac{\rho}{EI}} \tag{28}$$

For the comparison of the present numerical results, the cross-sectional dimensions ($b\times h$) of the nanobeams are taken as unity (h =1nm, b =1nm) and the length of the nanobeams is varied as (L = 5,10,20,100) nm.

FG nanobeam

Ceramic (Al_2O_3): E_c = 380 GPa, ρ=3800 kg/m^3 and μ= 0.3

Metal (Al): E_m =70 GPa, ρ=2700 kg/m^3 and μ= 0.3

$$\bar{w}=\frac{100E_mh^3}{q_0L^4}w,\quad \bar{N}=\frac{12L^2}{E_m h^3}N_0,\quad \bar{\omega}=\left(\omega L^2/h\right)\sqrt{\frac{\rho_m}{E_m}} \tag{29}$$

Table 1 presents the non-dimensional transverse deflections, critical buckling loads, and fundamental frequencies of isotropic nanobeams for various nonlocal parameters. The material properties and nondimensional form for isotropic nanobeams are given in Eq. (28). The present numerical results are compared with existing literature. Examination of Table 1 reveals that the present results are in close agreement with those presented by Thai [12], Thai and Vo [13], Thai *et al.* [14], and Aydogdu [18] using different higher-order nonlocal theories. The FSDT of Timoshenko [6] and the CBT of Bernoulli-Euler [5] underestimate the transverse deflection whereas overestimate the critical buckling load and fundamental frequencies due to neglect of the transverse shear deformation effect. It is also observed that the increase in the nonlocal parameter increases the transverse

Table 1. Non-dimensional transverse deflection, critical buckling load, and fundamental frequencies of simply supported isotropic nanobeams (L = 10 nm, h = 1 nm, b = 1 nm).

		Nonlocal parameter (n)				
Quantity	Theory	0	1	2	3	4
Transverse deflections ($\bar{W}$)	Present	1.3394	1.4675	1.5956	1.7237	1.8517
(m = 100)	Thai [12]	1.3346	1.4622	1.5898	1.7173	1.8449
	Thai and Vo [13]	1.3345	1.4621	1.5897	1.7173	1.8449
	Aydogdu [18]	1.3480	1.4921	1.6362	1.7802	1.9243
	Timoshenko [6]	1.3125	1.4383	1.5642	1.6900	1.8158
	Bernoulli-Euler [5]	1.3021	1.4271	1.5521	1.6771	1.8021
Critical buckling loads ($\bar{N}$)	Present	9.6230	8.7591	8.0372	7.4253	6.8993
(m = 1)	Thai [12]	9.6227	8.7583	8.0364	7.4244	6.8990
	Thai and Vo [13]	9.6231	8.7587	8.0367	7.4247	6.8994
	Aydogdu [18]	9.6242	8.7597	8.0377	7.4256	6.9001
	Timoshenko [6]	9.7891	8.9102	8.1753	7.5532	7.0181
	Bernoulli-Euler [5]	9.8701	8.9833	8.2433	7.6153	7.0752
Fundamental frequencies ($\bar{\omega}$)	Present	9.7131	9.2753	8.8843	8.5390	8.2316
(m = 1)	Thai [12]	9.7075	9.2612	8.8713	8.5269	8.2196
	Thai and Vo [13]	9.7077	9.2614	8.8715	8.5271	8.2198
	Thai et al. [14]	9.7454	9.2973	8.9059	8.5601	8.2517
	Timoshenko [6]	9.7889	9.3387	8.9459	8.5988	8.2887
	Bernoulli-Euler [5]	9.8289	9.3769	8.9818	8.6337	8.3229

Table 2. Non-dimensional transverse deflection, critical buckling load, and fundamental frequencies of simply supported isotropic nanobeams (h = 1 nm, b = 1 nm).

Quantity	L/h	Theory	Nonlocal parameter (n) 0	1	2	3	4
Transverse deflections ($\overline{W}$)	5	Present	1.4347	1.5716	1.7045	1.8415	1.9804
(m = 100)		Thai [12]	1.4320	1.5673	1.7027	1.8381	1.9735
	10	Present	1.3394	1.4675	1.5956	1.7237	1.8517
		Thai [12]	1.3346	1.4622	1.5898	1.7173	1.8449
	20	Present	1.3134	1.4406	1.5637	1.6919	1.8161
		Thai [12]	1.3102	1.4359	1.5615	1.6872	1.8128
	100	Present	1.3036	1.4301	1.5567	1.6802	1.8047
		Thai [12]	1.3024	1.4274	1.5525	1.6775	1.8025
Critical buckling loads ($\overline{N}$)	5	Present	8.9807	8.1703	7.5010	6.9244	6.4338
(m = 1)		Thai [12]	8.9519	8.1477	7.4761	6.9068	6.4181
	10	Present	9.6230	8.7591	8.0372	7.4253	6.8993
		Thai [12]	9.6227	8.7583	8.0364	7.4244	6.8990
	20	Present	9.8259	8.9415	8.2128	7.5767	7.0491
		Thai [12]	9.8067	8.9258	8.1900	7.5664	7.0310
	100	Present	9.8916	9.0066	8.2535	7.6235	7.0926
		Thai [12]	9.8671	8.9807	8.2405	7.6130	7.0743
Fundamental frequencies ($\overline{\omega}$)	5	Present	9.3016	8.8636	8.4996	8.1592	7.8644
(m = 1)		Thai [12]	9.2745	8.8482	8.4757	8.1466	7.8530
	10	Present	9.7131	9.2753	8.8843	8.5390	8.2316
		Thai [12]	9.7075	9.2612	8.8713	8.5269	8.2196
	20	Present	9.8474	9.3938	9.0075	8.6573	8.3451
		Thai [12]	9.8281	9.3763	8.9816	8.6328	8.3218
	100	Present	9.8874	9.4220	9.0241	8.6725	8.3689
		Thai [12]	9.8679	9.4143	9.0180	8.6678	8.3555

deflection but decreases the critical buckling load and fundamental frequency. The effects of various values of *L/h* on the transverse deflection, critical buckling load and the fundamental frequencies are presented in Table 2.

The material properties and nondimensional form for FG (Al_2O_3/Al) nanobeams beams are given in Eq. (29). Table 3 shows a comparison of non-dimensional transverse deflections of simply supported FG nanobeam subjected to uniform load. The numerical results are presented for various values of nonlocal parameter (n = 0, 1, 2, 3, 4) and the power-law coefficients (k = 0, 0. 5, 1, 5, 10). At k = 0, the nanobeam is ceramic-rich whereas, at $k = \infty$, the nanobeam is metal-rich. The authors have also generated numerical results using Reddy's TSDT [10], the FSDT [6], and the CBT [5] for comparison purpose. Table 3 shows that the present theory is in good agreement with Reddy's theory while predicting transverse deflections. Non-dimensional transverse deflection increases with an increase in the power-law coefficient as well as a nonlocal parameter. The variations of transverse deflection are also shown in Fig. 3(a) and Fig. 3(b).

Effect of nonlocal parameter and the power-law coefficients on the critical buckling load of simply supported FG nanobeam is presented in Table 4. For the comparison purpose, the critical buckling loads are also generated using Reddy's theory, the FSDT, and the CBT. From Table 4 it is observed that the present theory is in good agreement with Reddy's TSDT [10], and the FSDT [6] whereas the CBT [5] overestimates the critical buckling loads due to neglect of the transverse shear deformation. When the nanobeam is made up of purely ceramic, it undergoes a large critical buckling load. As the stiffness of the nanobeam decreases the non-dimensional critical buckling load also decreases. Similarly, as the nonlocal parameter increases the non-dimensional critical buckling load decreases. Fig. 4(a) and Fig. 4(b) show the variations of non-dimensional critical buckling load with nonlocal parameter and the power-law coefficient.

The effects of the nonlocal parameter and the power-law coefficient on the fundamental frequencies of simply supported FG nanobeam are presented in Table 5. Fundamental frequencies obtained using the present theory are compared

Table 3. Non-dimensional transverse deflections ($\overline{W}$) of simply supported FG nanobeams under uniform load (*L* = 10 nm, *h* = 1 nm, *m* = 100).

		Power-law coefficients (*k*)				
Theory	*n*	0	0.5	1	5	10
Present	0	2.9556	4.6467	6.0409	9.2826	10.2290
	1	3.2345	5.0839	6.6093	10.1542	11.1889
	2	3.5134	5.5211	7.1776	11.0258	12.1488
	3	3.7922	5.9583	7.7462	11.8976	13.1088
	4	4.0711	6.3955	8.3145	12.7692	14.0688
Reddy [10]	0	2.9501	4.5373	5.8959	9.02040	9.94030
	1	3.2322	4.9713	6.4599	9.88200	10.8892
	2	3.5142	5.4053	7.0240	10.7436	11.8381
	3	3.7963	5.8394	7.5880	10.7436	12.7871
	4	4.0783	6.2734	8.1521	12.4669	13.7360
Timoshenko [6]	0	2.9382	4.5203	5.8727	8.94790	9.84910
	1	3.2192	4.9526	6.4345	9.80340	10.7904
	2	3.5003	5.3850	6.9964	10.6588	11.7317
	3	3.7814	5.8173	7.5582	11.5143	12.6729
	4	4.0625	6.2496	8.1201	12.3698	13.6142
Bernoulli-Euler [5]	0	2.8783	4.5218	6.0413	9.66760	9.97800
	1	3.1546	4.9546	6.6465	11.2823	10.9778
	2	3.4309	5.3874	7.2516	12.8970	11.9777
	3	3.7072	5.8202	7.8567	14.5117	12.9775
	4	3.9835	6.2529	8.4618	16.1262	13.9772

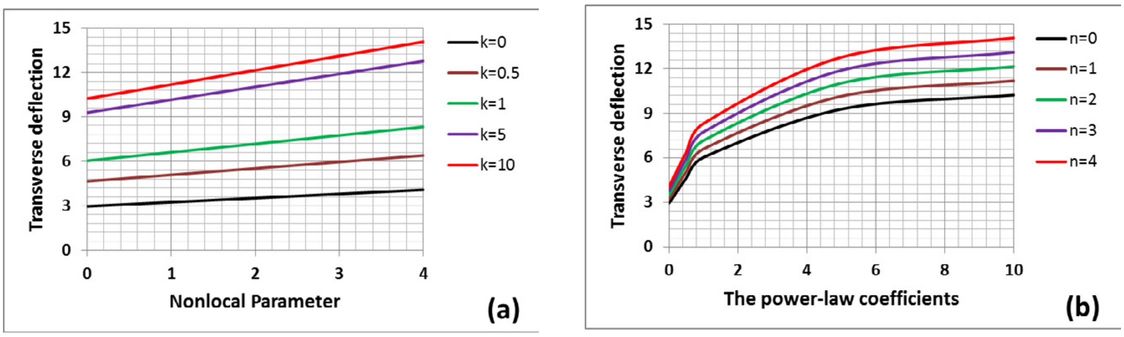

Fig. 3. The variation of the non-dimensional transverse deflection with (a) the nonlocal parameter (b) the power-law coefficient for simply supported FG nanobeam subjected to a uniform load (*L/h* = 10).

with those obtained using theories of Reddy [10], Timoshenko [6] and Bernoulli-Euler [5]. The present results are in good agreement with the TSDT of Reddy. Similar to critical buckling loads, the FSDT and the CBT also overestimate the fundamental frequencies for all values of nonlocal parameters and the power-law coefficients. Fig. 5(a) and Fig. 5(b) present the variation of fundamental frequency with the nonlocal parameter and the power-law coefficient.

Merits and demerits of the present study

The present study highlights the transverse deflection, critical buckling loads and the fundamental frequencies for isotropic and functionally graded nanobeams using a new non-polynomial type higher-order nonlocal beam theory. Merits and demerits of the present theory are summarized as below.

1) The present theory gives the realistic variation of transverse shear stress through-the-thickness of the beam satisfying boundary conditions at the top and bottom surfaces of the beam.
2) This theory does not require any shear correction factor to account for strain energy

Table 4. Non-dimensional critical buckling load ($\bar{N}$) of simply supported FG nanobeams (L = 10 nm, h = 1 nm, m = 1).

		Power-law coefficients (k)				
Theory	n	0	0.5	1	5	10
Present	0	51.9768	33.7972	26.0103	16.9966	15.4215
	1	47.3073	30.7614	23.6740	15.4693	14.0365
	2	43.4079	28.2262	21.7218	14.1947	12.8793
	3	40.1025	26.0770	20.0682	13.1131	11.8992
	4	37.2647	24.2312	18.6483	12.1858	11.0564
Reddy [10]	0	52.2328	33.9636	26.1384	17.0803	15.4975
	1	47.5402	30.9129	23.7906	15.5454	14.1056
	2	43.6216	28.3652	21.8288	14.2646	12.9427
	3	40.3000	26.2054	20.1670	13.1777	11.9578
	4	37.4483	24.3506	18.7401	12.2458	11.1109
Timoshenko [6]	0	52.2588	33.9806	26.1514	17.0893	15.5054
	1	47.5642	30.9279	23.8026	15.5534	14.1126
	2	43.6436	28.3792	21.8398	14.2716	12.9487
	3	40.3200	26.2184	20.1770	13.1847	11.9638
	4	37.4673	24.3626	18.7491	12.2518	11.1169
Bernoulli-Euler [5]	0	53.7996	34.9825	26.9223	17.5922	15.9624
	1	48.9661	31.8398	24.5045	16.0114	14.5285
	2	44.9305	29.2161	22.4838	14.6925	13.3307
	3	41.5088	26.9913	20.7719	13.5726	12.3168
	4	38.5721	25.0815	19.3021	12.6127	11.4439

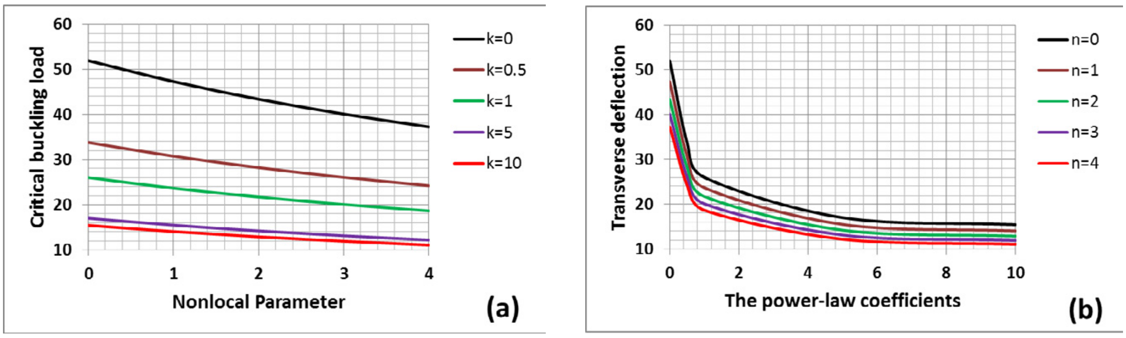

Fig. 4. The variation of the non-dimensional critical buckling load with (a) the nonlocal parameter (b) the power-law coefficient for simply supported FG nanobeam subjected to a uniform axial force (L/h = 10).

due to shear deformation.

3) A comparison of the numerical results through various tables show that the present theory is accurate and efficient for the analysis of nanobeams.
4) In the present study, the theory is applied for the analysis of simply supported boundary conditions of the beam. However, this theory can be extended for the analysis of other boundary conditions of the beam as well.

CONCLUSIONS

In this paper, an inverse trigonometric shear deformation theory is applied for bending, buckling, and free vibration analyses of functionally graded nanobeams. Eringen's nonlocal theory is used to capture the small size effects. The equations of motion are derived using Hamilton's principle. Analytical solutions are obtained using Navier's technique. The present results are compared with existing literature, Reddy's theory, the FSDT, and the CBT and agree well with those of higher order theories. It is concluded that the transverse deflections of nonlocal beams are greater than those of local beams, whereas nonlocal parameters reduce the buckling loads as well as fundamental frequencies. Based on the comparison of the numerical results obtained for various sizes of the nanobeams, various values of nonlocal parameter and the power-law coefficient

Table 5. Non-dimensional fundamental frequencies ($\bar{\omega}$) of simply supported FG nanobeams (*L* = 10 nm, *h* = 1 nm, *m* = 1).

Theory	*n*	Power-law coefficients (*k*)				
		0	0.5	1	5	10
Present	0	9.7072	8.2776	7.4848	6.4686	6.2665
	1	9.2620	7.8982	7.1421	6.1719	5.9797
	2	8.8720	7.5658	6.8408	5.9126	5.7280
	3	8.5274	7.2714	6.5746	5.6820	5.5050
	4	8.2202	7.0095	6.3381	5.4779	5.3069
Reddy [10]	0	9.7082	8.2784	7.4855	6.4692	6.2671
	1	9.2629	7.8990	7.1428	6.1725	5.9803
	2	8.8729	7.5666	6.8415	5.9132	5.7286
	3	8.5283	7.2721	6.5753	5.6826	5.5056
	4	8.2210	7.0102	6.3387	5.4784	5.3074
Timoshenko [6]	0	9.7161	8.2852	7.4916	6.4745	6.2722
	1	9.2705	7.9055	7.1486	6.1775	5.9852
	2	8.8802	7.5728	6.8471	5.9180	5.7333
	3	8.5353	7.2780	6.5807	5.6872	5.5101
	4	8.2277	7.0159	6.3439	5.4829	5.3117
Bernoulli-Euler [5]	0	9.8296	8.3819	7.5791	6.5501	6.3454
	1	9.3787	7.9977	7.2321	6.2497	6.0551
	2	8.9838	7.6612	6.9270	5.9871	5.8002
	3	8.6349	7.3630	6.6575	5.7536	5.5744
	4	8.3238	7.0978	6.4179	5.5469	5.3737

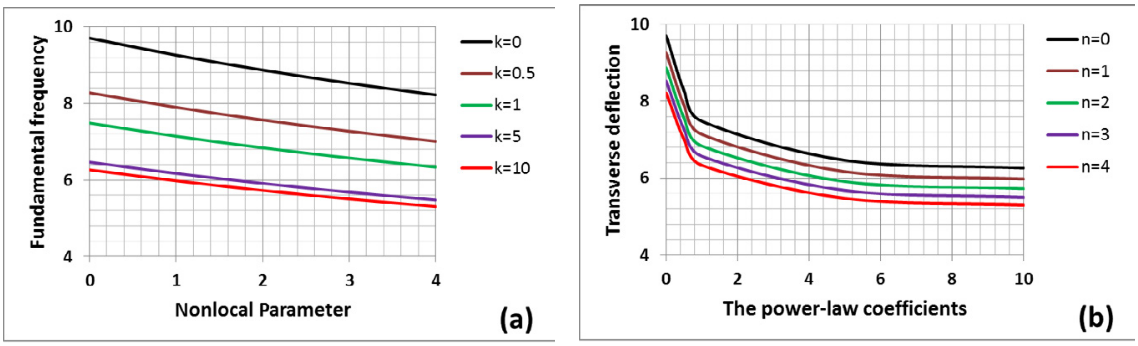

Fig. 5. The variation of the non-dimensional fundamental frequency with (a) the nonlocal parameter (b) the power-law coefficient for simply supported FG nanobeam (*L/h* = 10).

accounting for the gradation of material properties, finally it is concluded that the present theory is capable enough to capture the nano size effects of nanobeams as well as gradation of material properties.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

REFERENCES

[1] Eltaher M. A., Khater M. E., Emam S. A., (2016), Review on nonlocal elastic models for bending, buckling, vibrations, and wave propagation of nanoscale beams. *Appl. Math. Model.* 40: 4109–4128.

[2] Farajpour A., Ghayesh M. H., Farokhi H., (2018), A review on the mechanics of nanostructures. *Int. J. Eng. Sci.* 133: 231–263.

[3] Chandel V. S., Wang G., Talha M., (2020), Advances in modelling and analysis of nano structures: A review**.** *Nanotechnol. Rev.* 9: 230–258.

[4] Eringen A. C., (1983), On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *J. Appl. Phys.* 54: 4703-4710.

[5] Bernoulli J., (1694), Curvatura laminae elasticae. *Acta Eruditorum Lipsiae*. 3: 262–276.

[6] Timoshenko S. P., (1921), On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philos. Mag.* 41: 742–746.

[7] Sayyad A. S., Ghugal Y. M., (2015), On the free vibration analysis of laminated composite and sandwich plates: A review of recent literature with some numerical results. *Compos. Struct.* 129: 177–201.

[8] Sayyad A. S., Ghugal Y. M., (2017), Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature. *Compos. Struct.* 171: 486–504.

[9] Sayyad A. S., Ghugal Y. M., (2018), Modeling and analysis of functionally graded sandwich beams: A review. *Mech. Adv. Mater. Struct.* 26: 1776-1795.

[10] Reddy J. N., (2007), Nonlocal theories for bending, buckling and vibration of beams. *Int. J. Eng. Sci.* 45: 288-307.

[11] Nikam R. D., Sayyad A. S., (2020), A unified nonlocal formulation for bending, buckling and free vibration analysis of nanobeams. *Mech. Adv. Mater. Struct.* 27: 807-815.

[12] Thai H., (2012), A nonlocal beam theory for bending, buckling, and vibration of nanobeams. *Int. J. Eng. Sci.* 52: 56–64.

[13] Thai H., Vo T. P., (2012), A nonlocal sinusoidal shear deformation beam theory with application to bending, buckling, and vibration of nanobeams. *Int. J. Eng. Sci.* 54: 58–66.

[14] Thai S., Thai H-T., Vo T. P., Patel, V. I., (2018), A simple shear deformation theory for nonlocal beams. *Compos. Struct.* 183: 262–270.

[15] Darjani H., Mohammadabadi H., (2014), A new deformation beam theory for static and dynamic analysis of microbeams. *Int. J. Mech. Sci.* 89: 31–39.

[16] Hashemi S. H., Khaniki H. B., (2017), Vibration analysis of a timoshenko non-uniform nanobeam based on nonlocal theory: An analytical solution. *Int. J. Nano Dimens.* 8: 70-81.

[17] Arefi M., (2019), Magneto-electro-mechanical size-dependent vibration analysis of three-layered nanobeam with initial curvature considering thickness stretching. *Int. J. Nano Dimens.* 10: 48-61.

[18] Aydogdu M., (2009), A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration. *Physica E: Low Dimens. Syst. Nanostruct.* 41: 1651-1655.

[19] Bouazza M., Amara K., Zidour M., Tounsi A., Bedia E. A. A., (2015), Postbuckling analysis of nanobeams using trigonometric shear deformation theory. *App. Sci. Report* 10: 112-121.

[20] Mahmoud F. F., Eltaher M. A., Alshorbagy A. E., Meletis E. I., (2012), Static analysis of nanobeams including surface effects by nonlocal finite element. *J. Mech. Sci. Technol.* 26: 3555-3563.

[21] Roque C. M. C., Ferreira A. J. M., Reddy J. N., (2011), Analysis of Timoshenko nanobeams with a nonlocal formulation and meshless method. *Int. J. Eng. Sci.* 49: 976–984.

[22] Tounsi A., Semmah A., Bousahla A. A., (2013), Thermal buckling behavior of nanobeams using an efficient higher-order nonlocal beam theory. *J. Nanomech. Micromech.* 3: 37-42.

[23] Akgoz B., Civalek O., (2014), A new trigonometric beam model for buckling of strain gradient microbeams. *Int. J. Mech. Sci.* 81: 88–94.

[24] Akgoz B., Civalek O., (2012), Analysis of micro-sized beams for various boundary conditions based on the strain gradient elasticity theory. *Arch. Appl. Mech*. 82: 423–443.

[25] Bedia W. A., Houari M. S. A., Bessaim A., Bousahla A. A., Tounsi A., Saeed T., Alhodaly M. S., (2019), New hyperbolic two-unknown beam model for bending and buckling analysis of a nonlocal strain gradient nanobeams. *J. Nano Res.* 57: 175-191.

[26] Simsek M., Yurtcu H. H., (2013), Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory. *Compos. Struct.* 97: 378–386.

[27] Simsek M., Reddy J. N., (2013), Bending and vibration of functionally graded micro beams using a new higher order beam theory and the modified couple stress theory. *Int. J. Eng. Sci.* 64: 37-53.

[28] Akgoz B., Civalek O., (2015), Bending analysis of FG microbeams resting on Winkler elastic foundation via strain gradient elasticity. *Compos. Struct.* 134: 294-301.

[29] Lei J., He Y., Zhang B., Gan Z., Zeng P., (2013), Bending and vibration of functionally graded sinusoidal micro beams based on the strain gradient elasticity theory. *Int. J. Eng. Sci.* 72: 36-52.

[30] Ebrahimi F., Barati M. R., (2015), A nonlocal higher-order shear deformation beam theory for vibration analysis of size-dependent functionally graded nanobeams. *Arab. J. Sci. Eng.* 115: 41-50.

[31] Ansari R., Gholami R., Shojaei M. F., Mohammadi V., Sahmani S., (2013), Size-dependent bending, buckling and free vibration of functionally graded Timoshenko microbeams based on the most general strain gradient theory. *Compos. Struct.* 100: 385–397.

[32] Yu T., Hu H., Zhang J., Bui T. Q., (2019), Isogeometric analysis of size-dependent effects for functionally graded microbeams by a non-classical quasi-3D theory. *Thin Walled Struct.* 138: 1–14.

[33] Aria A. I., Friswell M. I., (2019), A nonlocal finite element model for buckling and vibration of functionally graded nanobeams. *Compos. Part B- Eng.* 166: 233–246.

[34] Reddy J. N., (2011), Microstructure-dependent couple stress theories of functionally graded beams. *J. Mech. Phys. Solids.* 59: 2382–2399.

[35] Eltaher M. A., Emam S. A., Mahmoud F. F., (2013), Static and stability analysis of nonlocal functionally graded nanobeams. *Compos. Struct.* 96: 82–88.

[36] Khaniki H. B., (2019), On vibrations of FG nanobeams. *Int. J. Eng. Sci.* 135: 23–36.

[37] Salamat-talab M., Nateghi A., Torabi J., (2012), Static and dynamic analysis of third-order shear deformation FG micro beam based on modified couple stress theory. *Int. J. Mech. Sci.* 57: 63–73.

[38] Zenkour A., Ebrahimi F., Barati M. R., (2019), Buckling analysis of a size-dependent functionally graded nanobeam resting on Pasternak's foundations. *Int. J. Nano Dimens.* 10: 141-153.

[39] Sayyad A. S., Ghugal Y. M., (2020), Bending, buckling and free vibration analysis of size-dependent nanoscale FG beams using refined models and Eringen's nonlocal theory. *Int. J. Appl. Mech.* 12: 1-34.

[40] Nguyen T. K., Vo T. P., Nguyen B. D., Lee J., (2016), An analytical solution for buckling and vibration analysis of functionally graded sandwich beams using a quasi-3D shear deformation theory. *Compos. Struct.* 156: 238–252.

[41] Sayyad A. S., Ghugal Y. M., (2018), An inverse hyperbolic theory for FG beams resting on Winkler-Pasternak elastic foundation. *Adv. Aircr. Spacecr. Sci. Int. J.* 5: 671-689.

[42] Wakashima K., Hirano T., Niino M., (1990), Space applications of advanced structural materials. *ESP. SP-303:397.*